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ELEMENTARY ALGEBRA:

DESIGNED AS

A FIRST BOOK OF ALGEBRA

FOR

ALL GRADES OF STUDENTS.

BY

THOMAS K. BROWN,
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NOTE.

THROUGHOUT its various stages this work has had the benefit of the suggestions and criticisms of Isaac Sharpless, Professor of Mathematics at Haverford College, whose Geometries have been prepared simultaneously with it. The authors have kept in view the object of making these books parts of a connected series of Mathematical text-books.

The following volumes are now completed:

ELEMENTARY ALGEBRA. By THOMAS K. BROWN.

ELEMENTARY PLANE GEOMETRY. By ISAAC SHARPLESS.

ELEMENTS OF PLANE AND SOLID GEOMETRY. By ISAAC SHARPLESS.

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PREFACE.

IN the following work the elementary principles of Algebra are derived and illustrated by a carefully arranged system of mental and written exercises. These exercises are generally devised to perform a double office. They give *practice* in the subject immediately in hand, while they introduce *methods* and suggest *principles* which are to follow. By this means definitions and principles are usually only generalizations of the preceding work, and require no special formal demonstrations. With the exception of the Binomial Theorem, which is amply illustrated and explained, no principle is introduced not easily derived from preceding work or simple explanation. The principles of Arithmetic are never extended to algebraic operations, the author believing that Algebra should explain Arithmetic, rather than depend upon it.

The **Mental Exercises** form an important and original feature. Nearly all the easier examples are given in this way, and are intended to be worked mentally before the recitation, and to be performed in the class-room, orally or mentally, with the book open before the student. On account of the little time required for their solution, more examples have been introduced than would have been possible were the slate required. They give the teacher a superior opportunity of testing the knowledge of his class, and of drilling them thoroughly on important and difficult points. By such exercises the student acquires a *facility* in mental operations which is essential to rapid work in all the more advanced part.

From the nature of the treatment there is an almost continual review. As soon as one operation is understood, its application is extended to everything that the student has previously learned. Thus, most of the subjects are presented two or more times. But each, somewhere, receives an exhaustive handling.

Questions on the explanations, principles, and methods of work have frequently been introduced. These are intended to call important points to the student's mind, and compel him to frame in words those reasons which he has understood, but which are liable to be forgotten if not carefully enunciated.

The answers to the examples are all given in an Appendix, which can be omitted when desired. Their absence from the pages gives the pupil greater self-confidence, and ensures more careful work, especially at the blackboard.

In all classes in Algebra some students are found to be slower than others, which is often a serious hinderance to the advancement of the class. It is believed that general uniformity of progress can be secured by the use of the present book. The arrangement and explanations are such as will give to the average student an unusually clear perception of the principles of the study, while the duller ones will obtain sufficient facility to perform the examples, and by the continual practice gain as much real understanding of the principles as they are capable of acquiring.

CONTENTS.

PART I.

FUNDAMENTAL OPERATIONS AND FACTORING.

SECTION PAGE

I. Principal Signs 9

II. Representation of Letters by Numbers 10

III. Collecting 11

IV. Equations 14

V. Collecting (continued) 17

VI. Products of Letters 20

VII. Definitions.—Collecting (continued) 23

VIII. Addition 26

IX. Meaning and use of 0 (called Zero) 29

X. Subtraction 31

XI. Transposition 34

XII. Construction of Equations 36

XIII. Statement and Solution of Problems 38

XIV. Review and Summary 45

XV. Multiplication by Monomials 48

XVI. Review of Addition 51

XVII. Multiplication by a Polynomial 53

XVIII. Use of the Parenthesis 56

XIX. Solution of Problems 62

XX. Equations containing Two Unknown Quantities 68

XXI. Exercises in Equations containing Two Unknown Quantities 74

XXII. Solution of Problems involving Equations containing Two Unknown Quantities 76

XXIII. Division by Monomials 79

XXIV. Arrangement of Terms 83

XXV. Review of Subtraction 84

XXVI. Division (continued) 87

SECTION PAGE
XXVII. Theorems 91
XXVIII. Use of Theorems in Multiplication 93
XXIX. Use of Theorems in Division 96
XXX. Factoring.—Monomial Factors 97
XXXI. Factoring.—Polynomial Factors 99
XXXII. Factoring.—Additional Theorems 102
XXXIII. General Method of Factoring 103
XXXIV. Review 105

PART II.

LEAST COMMON MULTIPLE.—FRACTIONS.

XXXV. Least Common Multiple of Monomials 107
XXXVI. Least Common Multiple of Polynomials 109
XXXVII. Fractions.—General Principles and Definitions 111
XXXVIII. Reduction of Fractions to their Lowest Terms 112
XXXIX. Reduction of Fractions to Whole or Mixed Quantities. 116
XL. Reduction of Entire and Mixed Quantities to Fractions. 118
XLI. Reduction of Fractions to the Least Common Denominator 121
XLII. Addition of Fractions 125
XLIII. Subtraction of Fractions 127
XLIV. Addition and Subtraction of Mixed Quantities 128
XLV. Multiplication and Division of Fractions 130
XLVI. Multiplication of Mixed Quantities 136
XLVII. Review Questions 138
XLVIII. Clearing Equations of Fractions 140
XLIX. Formation and Solution of Fractional Equations 142
L. Problems producing Fractional Equations 144

SIMPLE EQUATIONS.

LI. Simple Equations.—Definitions 150
LII. Simple Equations containing Two Unknown Quantities.—Elimination by Substitution 151
LIII. Elimination by Comparison 154
LIV. General Exercise in solving Simple Equations containing Two Unknown Quantities 156

SECTION PAGE
LV. Equations containing Three Unknown Quantities............ 160
LVI. Simple Equations containing any number of Unknown Quantities 165
LVII. Mental Exercise in the Statement of Problems............... 169
LVIII. Problems to be solved by Two or more Unknown Quantities. 170
LIX. Miscellaneous Problems to be solved by Simple Equations. 175
LX. Review.. 178

PART III.

POWERS, ROOTS, AND QUADRATIC EQUATIONS.

LXI. Involution of Monomials.. 179
LXII. Involution of Polynomials.................................... 183
LXIII. Binomial Theorem... 184
LXIV. Evolution of Monomials....................................... 189
LXV. Square Root of Polynomials.................................. 193
LXVI. Pure Quadratic Equations..................................... 195
LXVII. Solution of Equations containing the Double Sign....... 198
LXVIII. Pure Quadratic Equations containing Two Unknown Quantities... 201
LXIX. Problems producing Pure Quadratic Equations.......... 205
LXX. Adfected Quadratic Equations................................ 207
LXXI. Solution of Adfected Quadratic Equations................. 209
LXXII. Problems producing Quadratic Equations.................. 213
LXXIII. Reduction of Radicals.. 215
LXXIV. Addition and Subtraction of Radicals....................... 221
LXXV. Reduction of Radicals (continued)........................... 223
LXXVI. Fractional Exponents... 224
LXXVII. Reduction of Radicals to the same Degree................ 226
LXXVIII. Multiplication and Division of Radicals.................. 227
LXXIX. Fractional Exponents... 228
LXXX. Negative Exponents.. 229
LXXXI. Operations with Negative and Fractional Exponents... 233
LXXXII. Squares of Radicals.—Radical Equations................. 236
LXXXIII. Ratio and Proportion.. 240

REMARKS TO STUDENTS.

In working the examples, use the answers given in the Appendix only to find whether the work has been correctly done. After the first one or two examples in each exercise have been solved correctly, work several at a time before comparing the results with the answers. If any are wrong, do them again.

In solving the mental exercises, do not look at each answer in turn, but write down on slate or paper all the numbers of the examples in one set, and place opposite to them the results—obtained mentally if possible—and then compare the column with the column of printed answers.

Pay careful attention to all the reading-matter. Nothing is put in but what is of use in the work.

Try to understand the *reasons* for everything.

ELEMENTARY ALGEBRA.

PART I.

FUNDAMENTAL OPERATIONS AND FACTORING.

SECTION I.

PRINCIPAL SIGNS.

1. THE calculations performed by Algebra differ from those of Arithmetic in two important points; first, because in the former the simpler operations, such as Addition or Multiplication, are often *indicated* by **signs**; and secondly, because **letters** are used to represent numbers. The advantages of these are that they make the work easier and shorter.

2. Operations of **Addition** are indicated by the sign $+$, **plus**. **Subtraction** is indicated by the sign $-$, **minus**. **Multiplication** is indicated by the sign $\times$; **Division** by the sign $\div$; and an equality by $=$. These are the signs that are most frequently used.

$2 + 5$ is read 2 *plus* 5, and means 2 *added* to 5.

$6 - 3$ is read 6 *minus* 3, and means 6 *less* 3.

5×4 is read 5 *multiplied by* 4.

$14 \div 7$ is read 14 *divided by* 7.

$9 - 4 = 5$ is read 9 *minus* 4 *equals* 5, or 9 *minus* 4 *is equal to* 5.

SECTION II.

REPRESENTATION OF NUMBERS BY LETTERS.

Remark.—The examples in Part I. are intended to be performed mentally by the student in the class, with the book open before him, except where it is especially stated otherwise. In preparing for the recitation he can work them on the slate if necessary.

3. Let the letter a stand for the number 3. Then $a + a$ means $3 + 3$, or 3 added to 3, which equals 6. $a + a + a$ means $3 + 3 + 3$, which equals 9. $a + a + a + a = 3 + 3 + 3 + 3$, and $3 + 3 + 3 + 3 = 12$.

Now, let b stand for 4, then $b + b = 4 + 4$, and $4 + 4 = 8$.

In the same way let the following examples be done:

1. If b stands for 4, what does $b + b + b$ equal?
2. What does $b + b + b + b$ equal?
3. If c stands for 5, what does $c + c + c$ equal?
4. What does $c + c + c + c + c$ equal?

We may use two letters in the same example.

Example. If a stands for 3 and b for 4, what does $a + a + b$ equal?

OPERATION.

$$a + a + b = 3 + 3 + 4$$
$$3 + 3 + 4 = 10$$

5. What does $a + b + b$ equal?
6. What does $a + a + a + b$ equal?
7. If c stands for 5, what does $a + a + c$ equal?
8. What does $a + a + b + c$ equal?
9. What does $a + b + c + c$ equal?
10. What does $a + a + c + c$ equal?

SECTION III.

COLLECTING.

4. $a + a + a$ indicates 3 a's added together, and may therefore be more shortly expressed by $3a$, where the 3 means that there are so many a's added together. In the same way, $b + b + b + b + b$ indicates 5 b's added together, and is represented more conveniently by $5b$. These numbers, 3 and 5, and other numbers similarly placed before letters, are called **coefficients**, and show how many times that letter, or the number which the letter represents, has been taken and added together.

Since 5 b's added together give a result equal to 5 times one b, the expression $5b$ equals 5 times b, or if b stands for 4, $5b$ equals 5 times 4, or 20. Hence, to find the value of $5b$, it is easier to regard it as meaning 5 times 4 than five 4's added together. The multiplication is more easily performed than the addition. The operation of putting numbers in place of the letters which represent them is called **Substitution.**

Letting a, b, and c stand for the same numbers as before—that is, letting $a = 3$, $b = 4$, and $c = 5$—find the values of the following expressions:

1. $4a$.
2. $5b$.
3. $6a$.
4. $7b$.
5. $3c$.
6. $5a$.
7. $6c$.
8. $9b$.
9. $12a$.

These methods enable us to perform the examples in Art. 3 more briefly. Thus:

EXAMPLE. What does $b + b + b + b$ equal?

OPERATION.

$$b + b + b + b = 4b$$
$$4b = 16$$

EXAMPLE. What does $a + a + a + b + b$ equal?

OPERATION.

$$a + a + a + b + b = 3a + 2b$$
$$3a + 2b = 9 + 8$$
$$9 + 8 = 17$$

The student will perform the work in the class thus: $a + a + a + b + b$ equals $3a + 2b$. $3a + 2b$ equals $9 + 8$, and $9 + 8$ equals 17.

$$a = 3,\ b = 4, \text{ and } c = 5.$$

In the same way find what the following expressions equal:

1. $a + a + c + c + c$.
2. $b + b + b + c + c$.
3. $a + b + b$.
4. $a + a + b + c$.
5. $a + b + c + c$.
6. $a + b + b + c$.

Since a, b, and c can stand for any numbers, find the values of the following expressions, letting

$$a = 5,\ b = 3, \text{ and } c = 1.$$

ILLUSTRATION. $4a = 20$ and $7b + 8c = 29$.

1. $5a$.
2. $4b$.
3. $7c$.
4. $6b$.
5. $2a + 4b$.
6. $3b + 5c$.
7. $9a + b + c$.
8. $3a + 2b + 2c$.
9. $a + a + b + b + b$.
10. $a + a + c + c + c + c$.
11. $a + a + b + c + c$.
12. $b + b + b + c + c + c$.

5. The operation of placing the foregoing expressions in the shorter form is called **Collecting**.

In collecting $a + a + a + b + b$, since there are 3 a's and 2 b's, the result would have been the same had the like letters not been placed together. Thus, $a + a + a + b + b$, $a + a + b + a + b$, and $a + b + a + b + a$ are all equal to each other, for in each there are 3 a's and 2 b's added together, and collecting each would give us $3a + 2b$.

Also, $a+b+a+b+b=2a+3b$,
and $a+b+c+a+b+a=3a+2b+c$.

Collect the following. It is not necessary in these cases to find the numerical values:

1. $a+a+c+c+a$.
2. $a+b+b+b$.
3. $b+c+b+c$.
4. $a+a+b+c+b+b+a$.
5. $a+a+b+c+b+a+c$.
6. $a+b+c+a+c+b+a$.
7. $b+c+a+b+a+c$.
8. $c+a+b+b+c$.
9. $a+b+c+b+b+b$.
10. $a+b+c+c+b+c$.
11. $c+b+a+a+b$.
12. $a+a+c+c+c+a$.

We may use any other letters, and if we wish only to collect them, and do not care to find the numerical values of the expressions, it makes no difference if we do not know what numbers they represent.

Thus, $x+x+x+y+y=3x+2y$,
$a+a+x+y+x+a+y=3a+2x+2y$.

Collect the following, taking the letters in their alphabetical order:

1. $x+x+x$.
2. $a+a+a+y+y$.
3. $b+y+b+y$.
4. $a+a+x+a+x$.
5. $a+b+x+a+x$.
6. $b+c+c+y+c$.
7. $c+x+x+y+c$.
8. $x+y+a+x+a$.
9. $a+m+m+a$.
10. $c+x+a+x+c$.
11. $m+n+m+x+n+x$.
12. $x+y+x+x+y+a$.
13. $c+b+b+c+a+b$.
14. $a+x+x+a+x+x$.
15. $y+b+y+y+b+b$.
16. $m+n+m+x+m$.
17. $c+x+c+c+y+c$.
18. $a+y+y+a+b+a$.
19. $y+x+x+x+y+y$.
20. $c+m+n+c+c$.
21. $m+x+m+a+a$.
22. $c+a+x+y+a+m$.

6. Because $3a$ means 3 a's added together, and $5a$ means 5 a's added together, their sum will evidently give 8 a's; hence, $3a+5a=8a$. Also, for the same reason, $2b+3b=5b$, $6c+8c=14c$, $9a+2a=11a$, $5b+b=6b$.

Collect in this way the following:

1. $2a+7a$.	7. $3c+4c+2m$.
2. $5b+9b$.	8. $4a+2a+5c+2c$.
3. $4x+2x$.	9. $5b+2b+3x+6x$.
4. $5c+3c$.	10. $8m+3m+5y+4y$.
5. $9y+3y$.	11. $3c+c+y+y$.
6. $4m+3m$.	12. $4a+2a+a+3b+b$.

It makes no difference in what order the letters come. We wish to find how many there are altogether. Thus, $3a+2c+2a$ means 3 a's, 2 c's, and 2 more a's added together—that is, 5 a's and 2 c's—and therefore equals $5a+2c$.

Also, collect in this way the following:

1. $2a+3b+a+2b$.	4. $3a+3b+4a+2b+2c$.
2. $4c+2x+3c+7x$.	5. $8a+2x+3a+5x$.
3. $5b+2y+3y+6b$.	6. $7m+2y+3m+4y+m$.

SECTION IV.

EQUATIONS.

7. We have found the values of various expressions when we knew what numbers the letters stood for. It is more common, however, to have occasion to find the numbers that the letters stand for, when we know what the whole expression equals.

If x stand for 4, we find that $3x+2x=20$. Now, if we know that $3x+2x=20$, can we find what number x

stands for? We know that $3x + 2x = 20$. Collecting, we have $5x = 20$. Now, if $5x$'s $= 20$, one x, or simply x, must equal one-fifth of 20, or 4; we therefore simply say, Dividing by 5 we find $x = 4$.

8. The statement $3x + 2x = 20$ is called an **equation.** The word *equation* means literally, "*a making equal.*" The part of the equation on the left side of the equal mark is called the **first member** of the equation. The other part is called the **second member.**

In the next two examples the student will pay close attention to the explanations. Let other similar examples be explained in exactly the same manner, except where the teacher wishes all the work to be done mentally and only the result spoken.

Example. Find the value of x when

$$4x + 5x + 2x = 33.$$

Collecting the first member, we have $11x = 33.$

Dividing by 11, we find $x = 3.$

This result may be proved by finding what the expression $4x + 5x + 2x$ does equal when $x = 3$. We find $4x + 5x + 2x = 12 + 15 + 6$, and $12 + 15 + 6 = 33$. In the same way, we may always find whether we have the right value for the letter. We put this value in place of the letter in the first equation, and see whether the two members are equal. If they are not equal, some mistake has been made, and the work must be done again.

Finding the values of letters in an equation is called *solving the equation.*

In solving the preceding equation two operations were performed, viz.: *Collecting* and *Dividing by the coefficient of* x (or y).

EXAMPLE. Find the value of y when

$$2y + 3y + y = 7 + 4 + 1.$$

Collecting both members, we have $6y = 12.$

Dividing by 6, we find $y = 2.$

Proof. Substituting 2 for y in the first equation, we have

$$4 + 6 + 2 = 7 + 4 + 1.$$

Collecting, $12 = 12.$

The following equations may be solved both *mentally* and in *writing:*

1. Find the value of x when $6x + 2x = 32.$
2. Find the value of y when $2y + 5y = 21.$
3. Find the value of x when $4x + 3x = 56.$
4. Find the value of y when $7y + 3y = 62 + 8.$
5. Find the value of y when $y + 2y + 2y = 19 + 6.$
6. Find the value of x when $4x + x + 4x = 32 + 4.$
7. Find the value of y when $3y + 2y + 2y + y + 3y = 55.$
8. Find the value of x when $2x + 4x + 3x = 17 + 1.$
9. Find the value of x when $3x + x + 2x = 30 + 2 + 4.$
10. Find the value of y when $8y + 5y = 20 + 6.$
11. Find the value of y when $5y + 2y = 28 + 2 + 5.$
12. Find the value of y when $y + 2y + 2y = 1 + 16 + 3.$
13. Find the value of x when $4x + 2x + 3x = 9 + 6 + 3.$
14. Find the value of x when $10x + 5x = 41 + 4.$
15. Find the value of x when $20x + 4x = 40 + 8.$
16. Find the value of x when $2x + 2x + 3x = 1 + 20.$
17. Find the value of x when $9x + 2x = 90 + 1 + 8.$
18. Find the value of x when $5x + 2x + 2x = 52 + 2.$
19. Find the value of x when $3x + 2x + x = 30 + 6.$
20. Find the value of y when $9y + y + 2y = 8 + 40.$

SECTION V.

COLLECTING (Continued).

9. WHAT is the difference between $8a$ and $3a$?

It is evident that if we take 3 a's from 8 a's we will have 5 a's left. We here have subtracted $3a$ from $8a$, and since the sign $-$ indicates subtraction, we say $8a - 3a = 5a$. For the same reason $7b - 4b = 3b$. These operations are also called collecting.

EXAMPLES.

Collect in this way the following:

1. $6c - 4c$.
2. $9x - 5x$.
3. $7y - 2y$.
4. $8b - 6b$.
5. $9y - 7y$.
6. $10a - 4a$.
7. $8m - 3m$.
8. $12x - 9x$.

Let us now subtract both $3a$ and $2a$ from $7a$. We would have $7a$ less $3a$ and less $2a$, which we may express thus:

$$7a - 3a - 2a;$$

and since both $3a$ and $2a$, or $5a$, are to be taken from $7a$, there will be left $2a$.

We would perform the operation thus:

$$7a - 3a - 2a = 7a - 5a$$
$$7a - 5a = 2a$$

Collect $6c + 3c - 4c - 2c$.

EXPLANATION. Here both $4c$ and $2c$ are to be taken from the sum of $6c$ and $9c$.

OPERATION.

$$6c + 3c - 4c - 2c = 9c - 6c$$
$$9c - 6c = 3c$$

EXAMPLES.

In the same way collect the following:

1. $7c - 2c - 3c$.
2. $8x - 3x - x$.
3. $10y - 3y - 4y$.
4. $12m - 2m - 5m$.
5. $4a + 5a - 2a - 3a$.
6. $7y + 3y - 2y - 4y$.
7. $5b + 6b - 8b - 2b$.
8. $9y + 4y - 3y - 6y$.
9. $4b + 2b + 3b - 5b$.
10. $7a + 5a - 3a - 4a - 2a$.

It makes no difference in what order the parts come.

$3b + 4b - 2b$ and $3b - 2b + 4b$ both equal $5b$. Also,

$$6a - 2a + 4a - 3a = 10a - 5a$$
$$10a - 5a = 5a$$

11. $9y - 4y + 2y$.
12. $4m - 5m + 9m$.
13. $6b - 3b + 2b$.
14. $8a - 9a + 7a - 2a$.
15. $2c + 4c - 5c + 3c$.
16. $9y - 8y + 3y - 2y$.

Two or more letters may occur in the same expression. These are collected separately, because they stand for different numbers.

Ex. Collect $10c - 3b + 4c - 5c$.

OPERATION.

$$10c - 3b + 4c - 5c = 14c - 5c - 3b$$
$$14c - 5c - 3b = 9c - 3b$$

EXAMPLES.

Collect the following expressions:

1. $4a + 3b + 2a - 2b$.
2. $5c - 2a + 4a - 3c$.
3. $4c + 3c - 4b$.
4. $4b - 3c + 2b + 4c$.
5. $10b - 2a - 3a + 4b$.
6. $4c + 3a - 2b - 3b$.
7. $5a + 4a - 3a + b$.
8. $9x - 5y + 7y + 4x$.
9. $4b + 3a + 5b + 2a$.
10. $2m + 3x - x + 3m$.
11. $8a + 4b - 4a - 2b$.
12. $9y - 2y + 3x - y + 3x$.

Ex. If $a = 3$, what is the value of $2a + 5a - 3a$?

OPERATION.

$$2a + 5a - 3a = 4a$$
$$4a = 12$$

EXAMPLES.

What is the value of the following if $x = 5$?

1. $4x - 2x + x$.	4. $3x + 5x - 2x - x$.
2. $8x - 3x - 4x$.	5. $10x - 3x - x - 2x$.
3. $x + 5x - 2x - 2x$.	6. $3x + 2x - x - x$.

---

10. What does x equal when $4x - 2x = 7 + 5$?

OPERATION.

Collecting both members, $2x = 12$.

Dividing by 2, we find $x = 6$.

Proof. Substituting 6 for x
in the first equation, $24 - 12 = 7 + 5$.

Collecting, $12 = 12$.

Solve the equation, $6y + 3y - 2y = 36 - 1$.

OPERATION.

Collecting both members, $7y = 35$.

Dividing by 7, we find $y = 5$.

Proof. Substituting 5 for y
in the first equation, $30 + 15 - 10 = 36 - 1$.

Collecting, $35 = 35$.

Solve in *writing* the following

EQUATIONS.

1. $4x + 2x - 3x = 18$.
2. $9y - 6y + 2y = 39 - 4$.
3. $3x + 4x - x - 5x = 7 + 3$.

4. $8x - 2x + 4x - 5x = 22 + 3.$
5. $4x = 7 - 8 + 9.$
6. $10y - 2y - 4y = 3 + 9 - 4.$
7. $-2x + 4x + 3x = 30 - 2 - 3.$
8. $7y + 4y - 8y = 20 + 10 - 3.$
9. $12y - 9y + 3y = 25 - 1.$
10. $x + 5x - 2x = 8 + 30 - 10.$

SECTION VI.

PRODUCTS OF LETTERS.

11. If $b = 4$, $3b$ means 3 4's. Now, if $a = 3$, $3b$ could be expressed by ab, which would mean b taken a times, or a times b. With the above values it is evident that $ab = 12$. ab then means a times b, or a and b multiplied together, or $a \times b$. Therefore, when two letters are written together it means that the number represented by one is to be taken as many times as is indicated by the number represented by the other, and is therefore equal to the product of the two numbers.

If $c = 2$ and $m = 5$, $cm = 2 \times 5$, and $2 \times 5 = 10$.
If $a = 6$ and $x = 4$, $ax = 24$.

EXAMPLES.

1. If $b = 3$ and $y = 12$, what does by equal?
2. If $c = 7$ and $x = 9$, what does cx equal?
3. If $a = 8$ and $m = 4$, what does am equal?

12. As bm means b times m, so abm means a times bm. Hence, to find the value of abm we multiply the value of m by that of b, and the product by that of a. It makes no difference, however, in what order the factors are taken.

EXAMPLES.

What are the values of the following expressions when

$a = 5,\ b = 4,\ c = 2,$ and $x = 1$?

1. abc.
2. bx.
3. acx.
4. aa.
5. aab.
6. acc.
7. $abcx$.
8. $abcc$.
9. $abxx$.

13. Since $aaaa$ equals the product of 4 a's, it is usual to express such a product by writing a small 4 at the upper right-hand of the letter, thus a^4. This is read *a fourth*. The small 4 is called an exponent, and indicates that 4 a's are to be multiplied together. So b^5 is the same thing as $bbbbb$, and means the product of 5 b's.

Instead of writing aa, we write a^2, and call it *a square*, because it means a multiplied by a, and a quantity multiplied by itself gives its square. So aaa is written a^3, and is commonly called *a cube*. $aaaaaa$ is written a^6, and is called *a sixth.*

$aabbb$ is written a^2b^3, which is read *a square, b cube.*

$abbcccc$ is written ab^2c^4, which is read *a, b square, c fourth.*

$axxax$ is written a^2x^3, which is read *a square, x cube.*

Express the following quantities in this shorter way:

1. $aabcc$.
2. $abbxx$.
3. $aabbbx$.
4. $bbcxx$.
5. $bccxxx$.
6. $aammx$
7. $aaabbxxxyy$.
8. $aabbaacxc$.
9. $bcbcbyy$.
10. $aammammn$.

Question.—If $c = 2$, what is the numerical value of c^2?

OPERATION.

$$c^2 = c \times c$$
$$c \times c = 2 \times 2$$
$$2 \times 2 = 4$$

What is the value of c^3?

OPERATION.

$$c^3 = ccc$$
$$ccc = 8$$

If $a = 3$, what does a^2 equal? what does a^4 equal?

If $b = 4$, what does b^2 equal?

EXAMPLES.

What are the values of the following expressions when

$$a = 3,\quad b = 4,\quad \text{and}\quad c = 2?$$

1. a^2b.
2. ab^2.
3. a^2c.
4. abc.
5. a^2bc.
6. abc^2.
7. $5a^2$.
8. $7bc$.
9. $4c^3$.
10. $2ac^2$.
11. a^2c^2.
12. ac^3.
13. $a^2 + a$.
14. $ab - a^2$.
15. $c + c^3$.
16. $b^2 - b$.

14. If $a = 2$, what does $3a^2$ equal? what does $3a$ equal? These are not the same; $3a^2$ means 3 times the value of a multiplied by a. $3a$ means only 3 times the value of a. Hence, as a^2 and a do not at all represent the same number, we cannot collect them into one quantity. $3a^2 + 3a$ cannot be collected further, neither can $2ab + 3b$, unless we use numerical values. Hence, only those quantities can be collected which have their literal parts exactly alike. These are called **similar quantities.** Thus, $2a^2 + a + 3a^2 = 5a^2 + a$.

EXAMPLES.

Collect the following expressions:

1. $b^2 + 2b + 3b^2$.
2. $ab + 2b^2 + 3b^2$.
3. $ab + b^2 + 3ab + 2b^2$.
4. $abc + 3b^2 + 2abc + b^2$.
5. $3ax + 4ay + 2ax + ay$.
6. $4y^2 + y + 3y + y^2$.
7. $5x^2 + 2x^2 - 3x^2$.
8. $9a^2y + 3a^2y - 5a^2y$.
9. $7xy - 4xy + 4y^2 - 2y^2$.
10. $3a^2 + 4b - 2a^2 - 2b$.
11. $6x^2 + 3x - 2x^2 - 2x$.
12. $7m^2 - 4m - 3m^2 + 6m$.
13. $a^2 - 2b + 4b^2$.
14. $4c^3 + 3ac^3 - 2c^3 - ac^3$.

SECTION VII.

DEFINITIONS.—COLLECTING (Continued).

15. THE parts of an algebraic expression which are connected by the signs plus or minus are called **Terms**. Thus, in the expression $4a^2 + 2ab - b^2$ there are three terms, $4a^2$, $2ab$, and b^2.

16. Where an expression has one part—that is, has no sign, + or −, except before it—it constitutes one term, and is called a **Monomial**; as, a, $4ax$, and $-5x^2y^3z^4$.

17. When an expression has more parts than one, connected by plus or minus signs, it is called a **Polynomial**; as, $a - b$, $2mn + 3n^2$, and $a^3 - 3a^2x + 3ax^2 - x^3$.

18. A polynomial with two terms is called a **Binomial**; as, $a + b$, $3x - y$, and $6ax^3 + 12x^2y$.

19. A polynomial with three terms is called a **Trinomial**; as, $a^2 - 2ab + b^2$ and $3a^3x + a^2xy - xy^2$.

Other polynomials with more terms have no distinguishing names.

20. Let us now collect $4y - 7y$. It is very evident that 7 y's cannot be taken from 4 y's. But we can take 4 of the 7 y's from $4y$. This would still leave 3 y's to be subtracted, but with nothing to subtract them from. We indicate this result by writing a minus sign before the $3y$, thus: $-3y$; that is, $4y - 7y = -3y$.

If we subtract $4y$ from $7y$, there is left remaining of the *minuend* $3y$. If we are asked to subtract $7y$ from $4y$, after we have subtracted all we can, there is left of the *subtrahend* $3y$. Now we will indicate that this remainder is a part of the subtrahend by prefixing the sign − to it.

In the same way we will subtract the following:

Minuend,	9	4
Subtrahend,	4	9
	5	-5

If we take 4 from 9, there is left of the *minuend* a remainder of 5. If we try to take 9 from 4, there will be 5 of the *subtrahend* left unsubtracted, and it is marked -5.

EXAMPLES.

1. A boy goes to a store and buys a ball worth 15 cents. He finds on opening his pocketbook that there are only 12 cents in it. As he cannot take 15 from 12, he pays 12 cents, and still owes 3. We would express his financial condition by -3. That is, 3 cents must be taken, or subtracted, from his next money to pay his debt.

2. A farmer goes to town, and sells to one man 3 dollars' worth, buys of another 2 dollars' worth, sells to another 4 dollars' worth, and buys of another 7 dollars' worth. Before starting out again to settle he examines his account. How will he come out?

There are $+\$3 - \$2 + \$4 - \$7 = +\$7 - \$9$. $\$7 - \$9 = -\$2$. That is, he would be 2 dollars short. It does not mean 2 dollars less than nothing, but that 2 dollars must be paid from some other money.

3. The water in a certain reservoir rose in one hour 7 inches; it fell the next 5 inches, it rose 2 the third, and fell 6 the fourth. How much had it changed?

Calling the rise of water + and the fall −, we have

$$+7-5+2-6=+9-11$$
$$+9-11=-2.$$

That is, the water fell 2 inches.

4. A boy rows up stream 15 yards, then stops to rest and drifts down 12 yards, then rows again 10, but gets caught by the swifter current and is carried down 16. How far is he from his starting-place?

5. He pulls his boat up and starts on again. He first drifts down 7 yards, then rows up 12, then drifts down 8 and rows up 10. Where is he then?

21. A quantity or term which has a minus sign before it is called a minus or *negative* quantity or term. One which has a plus sign before it is called a plus or *positive* quantity or term. *Where no sign is expressed the sign + is understood.*

All quantities are regarded as either positive or negative. Positive terms are to be added to whatever other quantities they are attached, while negative terms are to be subtracted.

We cannot always tell from what a negative term is to be subtracted any more than a man can always tell with what he is going to pay his debts. We must, however, indicate the real character of the term.

22. *When there are several positive and negative terms to be collected, we add together the positive or added terms, and also the negative or subtracted terms. We then take the difference of these sums and give it the plus or minus sign, according as it is the remainder of the added terms or of the subtracted terms. That is, we give it the sign of the greater quantity.*

Thus, $5a - 8a = -3a, \quad 4x - 9x = -5x,$

$$10ay - 12ay = -2ay.$$

EXAMPLES.

Collect the following:

1. $3y - 7y.$
2. $7z - 9z.$
3. $8a - 3a.$
4. $4ay + 2ay - 8ay.$
5. $7m^2 - 9m^2 + 4m^2.$
6. $9a^3 + 3a^3 - 8a^3.$
7. $5amx - 8amx - 3amx.$
8. $12abc - 9abc - 3b.$
9. $7x^2y - 3x^2y + 4a - 6a.$
10. $8a + 4b - 6a - 7b.$
11. $5x^2 - 8x^2 - 4ab + 2ab.$
12. $-2ab^2 + 5ab^2 + 4b^2c - 9b^2c.$

SECTION VIII.

ADDITION.

23. Addition, in Algebra, is the operation of finding the sum of two or more *separate* algebraic expressions. It differs from **Collecting,** because that means uniting the terms of *one* expression into their smallest possible number. Thus we say, *Collect* $4a - 2a$, or $3ax - 2x^2 + 4ax + 5x^2$; but we say, *Add* $4a$ and $-2a$, or add $3ax - 2x^2$ and $4ax + 5x^2$. These last would be written out thus:

$$\begin{array}{lrcr} & 4a & \qquad & 3ax - 2x^2 \\ & \underline{-\,2a} & & \underline{4ax + 5x^2} \\ \text{Sum,} & 2a & & 7ax + 3x^2 \end{array}$$

Where there are several expressions to be added, they are usually placed under each other, with the similar terms in the same column. These are then added together precisely as they would have been collected had they been in one expression. The results of the separate additions connected by the proper signs give the whole sum of the expressions.

EXAMPLES.

Add the following:

$$\begin{array}{lr} 1. & -4a \\ & 6a \\ & -2a \\ & \underline{3a} \end{array} \qquad \begin{array}{lr} 2. & -7ax \\ & 8ax \\ & 3ax \\ & \underline{-6ax} \end{array} \qquad \begin{array}{lr} 3. & 4ab \\ & -2ab \\ & 4ab \\ & \underline{-7ab} \end{array} \qquad \begin{array}{lr} 4. & 6m^2n \\ & -4m^2n \\ & -3m^2n \\ & \underline{-5m^2n} \end{array}$$

$$\begin{array}{lr} 5. & 3xy + 2y^2 \\ & -2xy - 3y^2 \\ & \underline{4xy - 3y^2} \end{array} \qquad \begin{array}{lr} 6. & 4ax + 3x^2 \\ & ax + 2x^2 \\ & \underline{-3ax - 4x^2} \end{array} \qquad \begin{array}{lr} 7. & 5a^2xy - 5x^2y^2 \\ & -3a^2xy + 4x^2y^2 \\ & \underline{a^2xy - 2x^2y^2} \end{array}$$

Although it is generally best to put in one column the terms which are similar in regard to their letters, yet they do not always come so arranged. We must then select out the similar ones and collect them as before.

Thus,

$$\begin{array}{c} 3a \\ 4b \\ \hline 3a+4b \end{array} \qquad \begin{array}{r} 5x \\ 3y \\ -2x \\ \hline 3x+3y \end{array} \qquad \begin{array}{r} 4a+3b \\ 2b-2a \\ \hline 2a+5b \end{array}$$

EXAMPLES.

Add the following:

$$1.\ \begin{array}{r} m^2 \\ 4m \\ -3m \\ 5m^2 \\ \hline \end{array} \qquad 2.\ \begin{array}{r} 3c^2d \\ -4cd \\ 5c^2d \\ 8cd \\ \hline \end{array} \qquad 3.\ \begin{array}{r} 8y^2 \\ -3y \\ -6y \\ -5y^2 \\ \hline \end{array} \qquad 4.\ \begin{array}{r} 7xz \\ 4z^2 \\ -5xz \\ 4z^2 \\ \hline \end{array} \qquad 5.\ \begin{array}{r} 3a^2y \\ 3ay^2 \\ -2a^2y^2 \\ -4ay \\ \hline \end{array}$$

$$6.\ \begin{array}{r} x+y \\ y+x \\ \hline \end{array} \qquad 7.\ \begin{array}{r} 4x^2+3y^2 \\ 2x-2x^2 \\ \hline \end{array} \qquad 8.\ \begin{array}{r} 4a^2+3ab-2b^2 \\ -2bc+3a^2-ab \\ \hline \end{array}$$

$$9.\ \begin{array}{r} 2a+7c+4m \\ m-a+3c \\ -5c+2m-4a \\ \hline \end{array} \qquad 10.\ \begin{array}{l} 5x^2+3x-2 \\ 4-3x^2+2x \\ -6+x^2 \\ \hline \end{array} \qquad 11.\ \begin{array}{l} 2am+3m^2 \\ 4m^2+2bm^2 \\ 5bm^2 \\ \hline \end{array}$$

$$12.\ \begin{array}{r} 5b^2cx+4c^2x+y^2 \\ -7b^2cx-c^2x+5y^2 \\ 4b^2cx+3c^2x \\ -b^2cx-4c^2x-4y^2 \\ \hline \end{array} \qquad 13.\ \begin{array}{r} 5m^2+6mn+nx \\ -2m^2+3mn+2nx \\ -4m^2-5mn-2nx \\ 3m^2-2mn-4nx \\ \hline \end{array}$$

Let the following be performed in writing:

14. Add $7my^2+3a^2x-4m^2+3$, $2a^2x+3m^2-4+5my^2$, and $-6-4m^2y+2my^2-9a^2x$.

15. Add $4abc^2+8ab^2c-6a^2bc$, $-3abc^2+2ab^2c+4a^2bc$, and $2abc^2-7ab^2c-4a^2bc$.

16. Add $6m^3n - 4m^2n^2 - 8mn^3$, $-5m^3n + 7m^2n^2 + 6mn^3$, and $2m^3n - 2m^2n^2 + mn^3$.

17. Add $12x^3 - 5x^2y + 6y^3$, $-7x^3 - 2x^2y - 6y^3$, $4x^3 - 5x^2y + 7y^3$, and $-3x^3 + 4x^2y - 9y^3$.

18. Add $-2b^2c^3x + 4b^3c^2 - 3b^4c + 5x^2$, $5b^2c^3x - 2b^3c^2 - 7b^4c - 2x^2$, $6b^2c^3x - 4b^3c^2 - 2b^4c - 6x^2$, and $b^2c^3x + 3b^3c^2 + 4b^4c - 8x^2$.

19. Add $9a^3 + 7x^2 - 3z$, $5x^2 - 4a^3 + 2z$, $6a^3 - 4c - 3x^2$, and $5x^2 + 3y + 2a^3$.

20. Add $2abcx^2 - 3a^2bcx + 2ab^3c$, $5a^2bcx + 4abcx^2 - 3ab^3c$, and $9ab^3c - 11abcx^2 + 8a^2bcx$.

24. Addition in Algebra we see to be different from addition in Arithmetic. In Arithmetic the simple sum is wanted; there are no signs to be considered. But we may have four combinations of signs in Algebra:

I.	II.	III.	IV.
Both signs +.	Both signs −.	The sign of the greater +.	The sign of the greater −.
$+7x$	$-5a$	$+8c$	$-9y$
$+3x$	$-2a$	$-3c$	$5y$
$+10x$	$-7a$	$5c$	$-4y$

For Addition in Algebra we have therefore the general

Rule. — *When the signs of similar terms are alike, add their coefficients, and give to the result the same sign that both terms had before. When the signs are different, find the sum of the positive terms, and also of the negative terms, and take the difference of the two coefficients of these sums, and give it the sign of the greater.*

In algebraic addition, therefore, we have to perform arithmetical subtraction as well as arithmetical addition.

SECTION IX.

MEANING AND USE OF 0 (called Zero).

25. THE symbol 0 is called **Zero** in the higher mathematics. In Arithmetic it is often called Naught. Zero is, however, preferable. It has two meanings:

I. When it stands by itself, as 0, or $4 - 0$, it denotes *absence of numerical value.* It may be said to mean *no number* or *none.* It seldom means *nothing*, and should never be so called.

II. When it is used with other quantities in the same term, as $0\,ax$, or $3\,a^2x^0$, it means *not any*, or *no.* Thus, $0\,ax$ means *not any ax*, and may be read *no ax.*

ILLUSTRATIONS. If a farmer has 7 sheep, and sells 4 to one man and 3 to another, how many has he left?

SOLUTION. 7 sheep − 4 sheep − 3 sheep = 0 sheep (read *no* sheep), or, $7 - 4 - 3 = 0$ (read zero).

The farmer, therefore, has *no* sheep, or *none.* But we cannot say that he has *nothing*, for he may have other animals.

If a boy finds 3 chestnuts and eats 2, then finds 5 and eats 6, how many has he?

SOLUTION. 3 nuts − 2 nuts + 5 nuts − 6 nuts = 8 nuts − 8 nuts = 0 nuts,

or, $3 - 2 + 5 - 6 = 0.$

The first result, 0 nuts, means *no* nuts; the second result, 0, means *none.* We cannot say that the boy has *nothing*, for he may have other things.

If $a = 5$, $3\,a$ equals 5 taken 3 times, which gives as a product 15. But $0\,a$ equals 5 taken *no* times, which gives *no* product, or 0. Therefore, $0\,a = 0$ (read *no* a = zero).

$5\,y + 0 = 5\,y$; that is, $5\,y$ increased by *no* number $= 5\,y$.

$3\,ax - 0\,ax = 3\,ax$; that is, $3\,ax$ diminished by $0\,ax = 3\,ax$.

$9\,a - 9\,a = 0$; that is, $9\,a$ diminished by $9\,a = 0\,a = 0$.

EXAMPLES.

Collect the following expressions:

1. $3x + 0.$
2. $0y + 4y.$
3. $0 - 7x^2.$
4. $-3y - 0.$
5. $7x^3 - 0x^3.$
6. $5x - 5x.$
7. $-7y + 4y + 3y.$
8. $4xy - 4xy.$
9. $4x^2 + 3x - 4x^2.$
10. $7ab - 2b^2 - 7ab.$
11. $3y - 3y + 4y.$
12. $-8ax + x^3 + 8ax.$

26. The sum of two equal quantities with contrary sign, as $6ax$ and $-6ax$, is 0, and the two quantities may be cancelled in any expression or addition example in which such occur. The symbol 0 is not expressed unless all the terms cancel out. This is illustrated in the following additions:

$$\begin{array}{lrrrrr}
 & -4y & 8a & 5x+2y & -2ab+c & 3a-x \\
 & \underline{4y} & \underline{-3a} & \underline{3x-2y} & \underline{2ab-5c} & \underline{-3a+x} \\
\text{Sum,} & 0 & 0 & 8x & -4c & 0
\end{array}$$

EXAMPLES.

Add the following:

$$\begin{array}{ccccc}
13. & 14. & 15. & 16. & 17. \\
ax & 7xy & -4y^3 & 8ab & 6ay-3y^2 \\
\underline{-ax} & \underline{-7xy} & \underline{4y^3} & \underline{-8ab-b^3} & \underline{2ay+3y^2}
\end{array}$$

$$\begin{array}{ccc}
18. & 19. & 20. \\
3x+4y & 3a^2-4ab-2b^2 & -2mn-4n^2+nx \\
2x-y & -2a^2+3ab+5b^2 & -4mn+3n^2-8nx \\
\underline{x-3y} & \underline{-a^2+3ab-3b^2} & \underline{6mn+n^2+2nx}
\end{array}$$

Ex. **What quantity must be added to $-5y^2$ to give 0?**

SOLUTION. $5y^2$, because $-5y^2 + 5y^2 = 0$.

To give 0, what quantity must be added to each of the following?

21. $3ax.$
22. $-8xy.$
23. $5y^2.$
24. $-10ab.$
25. $7xy^2.$
26. $-4x^2y.$
27. $6abc.$
28. $9ax^3.$

Ex. What two quantities must be added to $-4ay$ to give first 0, then $3ay$?

Solution. Adding $4ay$ to $-4ay$ gives 0. Adding $3ay$ to 0 gives $3ay$. Hence, we must add $4ay$ and $3ay$.

Ex. What two quantities must be added to $-9b$ to give first 0, then $-3b$?

Solution. Adding $+9b$ to $-9b$ gives 0. Adding $-3b$ to 0 gives $-3b$. Hence, we must add $9b$ and $-3b$.

In the same way perform the following:

29. What two quantities added to $10xy$ will give $4xy$?
30. To $-3x^2$ will give $7x^2$?
31. To $6ab^2$ will give $-8ab^2$?
32. To $-4bc^3$ will give $-7bc^3$?
33. To $2x^3$ will give $10x^3$?

27. We notice that in all of these cases *the quantities which are added are the* **former quantity with its sign changed,** *and the* **latter quantity.**

Note.—A sign is said to be changed when a + sign is put in place of a − sign, or a − sign in place of +.

SECTION X.

SUBTRACTION.

28. Subtraction is the process of finding the difference between two quantities.

One quantity differs from another by that quantity which must be added to the second to give the first.

Thus, 5 differs from 2 by 3, because 3 must be added to 2 to give 5. So 4 differs from 6 by -2, because 2 must be subtracted from, or -2 added to, 6 to give 4.

The first quantity is called the **Minuend**, the second the **Subtrahend**, and the difference is called the **Remainder.**

The **Remainder**, therefore, is that quantity which must be added to the **Subtrahend** to give the **Minuend.**

Now, we found in the last article that the most convenient quantities to add to one quantity, in order to make it equal another, was *the former quantity with its sign changed*, and *the latter quantity.*

Therefore, that which must be added to the subtrahend to get the minuend must equal *the sum of the subtrahend, with its sign changed*, and *the minuend.* Hence the

Rule.—In order to find the remainder change the sign of the subtrahend, and add it to the minuend.

EXAMPLES.

Minuend,	$5y$		$5y$
Subtrahend,	$\underline{-4y}$	The remainder is found by adding	$\underline{+4y}$
Remainder,	$9y$		$9y$
Minuend,	$-3x$		$-3x$
Subtrahend,	$\underline{-9x}$	The remainder is found by adding	$\underline{+9x}$
Remainder,	$6x$		$6x$
Minuend,	$-6ax$		$-6ax$
Subtrahend,	$\underline{4ax}$	The remainder is found by adding	$\underline{-4ax}$
Remainder,	$-10ax$		$-10ax$

29. It is not usual to write out the example a second time. We merely conceive the sign of the subtrahend to be changed, and add the quantities.

Ex. Subtract

Minuend,	$8y^3$
Subtrahend,	$\underline{-5y^3}$

EXPLANATION.—We conceive the sign of the subtrahend to be changed to plus, and add the two quantities, getting $13y$ for the remainder.

EXAMPLES.

Subtract the following, explaining as in the preceding example:

1. $\begin{array}{r} 7ax^2 \\ \underline{5ax^2} \end{array}$

2. $\begin{array}{r} -9am^3 \\ \underline{11am^3} \end{array}$

3. $\begin{array}{r} -7x^2y^4 \\ \underline{4x^2y^4} \end{array}$

4. $\begin{array}{r} 9x^3y \\ \underline{-6x^3y} \end{array}$

5. $\begin{array}{r} -3x^3y \\ \underline{-8x^3y} \end{array}$

6. $\begin{array}{r} 6abx^2 \\ \underline{-8abx^2} \end{array}$

7. $\begin{array}{r} 10abc \\ \underline{-8abc} \end{array}$

8. $\begin{array}{r} 4ax \\ \underline{ax} \end{array}$

9. $\begin{array}{r} -x^6 \\ \underline{10x^6} \end{array}$

10. $\begin{array}{r} 3b^2c^3 \\ \underline{5b^2c^3} \end{array}$

11. $\begin{array}{r} ax \\ \underline{4ax} \end{array}$

12. $\begin{array}{r} 14a^3y \\ \underline{-a^3y} \end{array}$

Where there are several connected quantities to be subtracted, we change the sign of each term in the subtrahend and add it to the similar term in the minuend.

Ex. Subtract $\begin{array}{r} 4ab - 3a^2 + b^2 \\ \underline{2ab + 3a^2 + 4b^2} \end{array}$

Explanation. Conceiving all the signs of the subtrahend to be changed, and adding, we have for the remainder,

$$2ab - 6a^2 - 3b^2.$$

13. $\begin{array}{r} 5by^2 - 3b^3 \\ \underline{2by^2 - 2b^3} \end{array}$

14. $\begin{array}{r} 7x^3y - 4x^2y^2 + 6yz \\ \underline{5x^3y + 3x^2y^2 + 8yz} \end{array}$

15. $\begin{array}{r} 12a^2 + 8b^2 - 4bc \\ \underline{15a^2 - 2b^2 - 7bc} \end{array}$

16. $\begin{array}{r} 4c^3n + 3c^2n^2 - 5n^3 \\ \underline{3c^3n + 3c^2n^2 + 5n^3} \end{array}$

17. $\begin{array}{r} 2axy - 5xy^2 + 4y^3 \\ \underline{3axy + 2xy^2 + 2y^3} \end{array}$

18. $\begin{array}{r} -3m^2 + 4mn + 2mn^2 \\ \underline{-8m^2 - 2mn + 9mn^2} \end{array}$

19. $\begin{array}{r} -5b^2c + 3bc^2 - mn \\ \underline{4b^2c - 5bc^2 - 5mn} \end{array}$

20. $\begin{array}{r} 7x^3 + 3x^2 - 4x \\ \underline{7x^3 - 3x^2 - 4x} \end{array}$

Note.—If the similar terms are not under each other, we must still collect them from the different parts of the examples.

21. $\begin{array}{l} 5a - 3b + 4c \\ \underline{2b - 3c + 4a} \end{array}$

22. $\begin{array}{l} 5x^2 + 3x + y \\ \underline{5x^2 - 2y} \end{array}$

23. $\begin{array}{l} 8xy + 9y^2 - 4z \\ \underline{3xy - 2z} \end{array}$

24. $\begin{array}{l} 5ax + 2x^2 \\ \underline{3ax - 4x^2 + xy} \end{array}$

25. $\begin{array}{l} 5ax - 3ay + 3az \\ \underline{4ax - 2x^2 + 3ay} \end{array}$

26. $\begin{array}{l} 3by^2 + 4b^2y - 3b^3 \\ \underline{7by^2 + 2b^2y + 5b^4} \end{array}$

SECTION XI.

TRANSPOSITION.

30. The equations which have been solved before have had all the letters on one side and all the numbers on the other, as, $4x - 3x + 2x = 17 + 1$. But equations often occur like this:

$$3x - 4 = 2x + 6.$$

This cannot be collected as it stands. We cannot collect $3x$ and -4. We will therefore try to get rid of the -4. Now, bear clearly in mind that the value of the part or member before the equal-mark is just the same as that of the part or member after it. If then we were to take away -4 from the first member only, it would remove it from the $3x$. But to keep the equality of values, we must also take -4 from the second member. We will do this, according to the rule for Subtraction, by changing the sign of the quantity to be taken away or subtracted, and adding or annexing it to the second member. We therefore add $+4$ to the second member.

We then have $3x = 2x + 6 + 4$. But we cannot collect $2x + 6 + 4$. We will therefore take away from the second member the $2x$. But to preserve the equality of values of the two sides, we must also take $2x$ away from the other member, or change its sign and annex it.

We then have $3x - 2x = 6 + 4$.

Collecting both members, we find $x = 10$.

31. It will be noticed above that when a quantity is removed from one member of an equation, it must also be added to the other member, with its sign changed. By this operation the quantity is simply placed across the equal-mark, and has its sign changed. This process is called **Transposition,** which means *placing across.*

Rule.—*Any quantity may be taken from one member of an equation to the other if its sign be changed.*

Thus, if $5x + 3 - 2 = 4 - 7x$,

transposing, we have $5x + 7x = -3 + 2 + 4$.

If $6 - 3y + 1 = 4y - 8y + 9$,

transposing, we have $-3y - 4y + 8y = -6 - 1 + 9$.

Transpose the following, so that all the terms containing letters may be in the first member, and all the rest in the second:

1. $8z - 2 + 4 = 12 + 4z - 2$.
2. $5y + 4x - 2y = y + 7$.
3. $2x - 4 - 3x = 1 - 8x + 9$.
4. $3x + 2 = 5 - 2y$.
5. $4 - 7 = 5 - 6x + 2x$.
6. $5 = 7 - 4x + 10 - 2x$.

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32. Given $4x - 3 = 2x + 9$, to find the value of x.

Transposing, we have $4x - 2x = 9 + 3$.

Collecting both members, $2x = 12$.

Dividing by 2, we find $x = 6$.

NOTE.—In explaining, be careful always to mention the operations of *Transposing*, *Collecting*, and *Dividing*.

Solve the following equations in writing. As a Mental Exercise, let the student make the necessary transpositions.

1. $5y - 3 + 2y = 12 + 4y$.
2. $6 + 8x = 14 + 2x + 4$.
3. $12x - 9 = 24 + x$.
4. $20 - 7y + 3 = -4y + 39 - 5y$.
5. $3y - 7 = 8 - 2y$.
6. $9x + 21 + 2x = 5x + 45$.
7. $7x - 8 = 5x + 40$.
8. $3 + 2x - 11 = -2$.
9. $8x - 3 + 7x = 4x + 5 + 10x$.
10. $12y - 9 + 8 = 14y - 3$.
11. $3 - 5x = 24 - 8x$.

SECTION XII.

CONSTRUCTION OF EQUATIONS.

33. Ex. If $x + 3$ be equal to 4 less than $2x$, what number does x represent?

Solution. If $x + 3$ be equal to 4 less than $2x$, the relation will be expressed in the equation,

$$x + 3 = 2x - 4.$$

Transposing, $x - 2x = -3 - 4$.

Collecting, $-x = -7$.

Changing signs, we find $x = 7$.

Proof. $x + 3 = 10$; $2x = 14$. 10 is 4 less than 14.

Note.—We do not substitute in the equation, for we want to see whether the equation itself is correctly formed.

Ex. If $3x - 6$ is 4 more than x, what is the value of x?

SOLUTION. Since $3x - 6$ is 4 more than x, $3x - 6$ must equal $x + 4$. Hence, we have the equation, $3x - 6 = x + 4$.

This can be solved and proved as before

In the same way solve the following

WRITTEN EXAMPLES.

***Rule.**—Find an equality between the quantities from the given relations, express this by an equation, and then solve the equation.*

As a Mental Exercise, let the student give the equations that result in each example.

1. If $3x - 2$ be 1 more than twice x, what number does x stand for?

2. If the sum of $x - 7$ and $2x - 10$ be equal to 10, what is the value of x?

3. If the sum of $x + 4$ and $x - 2$ be 6 less than $3x$, what is the value of x?

4. If x and 1 more than twice x be together equal to 19, what is the value of x?

5. If $5x$ be 1 more than the sum of x, $x + 1$, and $2x - 3$, what is the value of x?

6. If 6 more than x be 2 less than 3 times x, what does x represent?

7. If the sum of x, $x - 12$, and $x + 24$ be 7 times x, what is the value of x?

8. If 3 times a certain number, plus 2, equal 2 times the number, plus 8, what is the number?

SUGGESTION.—Let x represent the number, then $3x + 2 = 2x + 8$.

9. If 5 times a certain number, minus 9, equal the sum of twice the number and 3, what is the number?

10. If 7 less than twice a number be 8 more than the number, what is the number?

11. If 3 times a number added to 4 times the number, plus 9, be 10 less than 8 times the number, what is the number?

12. If 2 more than 4 times a number be 10 less than 5 times the number, what is the number?

13. 36 minus a certain number is 3 more than twice the number. What is the number?

14. 8 more than twice a certain number is less than 26 by 4 times that number. What is the number?

15. A number plus 10 is 4 less than 3 times itself. What is the number?

16. A number increased by 3 less than 5 times itself is equal to 29 diminished by twice itself. What is the number?

SECTION XIII.

STATEMENT AND SOLUTION OF PROBLEMS.

34. ONE of the uses of Algebra consists in solving problems the solution of which by Arithmetic would be long and difficult, or perhaps impossible. The first problems here given may be easily solved by Arithmetic, but they prepare the way for others in whose solution Algebra is very useful.

PROBLEM 1. John has a certain number of cents, and James has twice as many. Both together have 24 cents. How many has each?

SOLUTION. We will let x represent the number of cents which John has. Then will $2x$ represent the number which James has, because he has twice as many as John; and since both together have 24 cents, x and $2x$ added together must equal 24. We can express this equality by the equation

$$x + 2x = 24.$$

This equation can be readily solved.

Collecting, we have $3x = 24,$

Dividing by 3, we find $x = 8,$ which is the number of cents John had, because x represented that number,

and $2x = 16,$ which is the number which James had, because $2x$ represented his number.

Proof. $8 + 16 = 24.$

PROB. 2. One farmer has a certain number of sheep, and his neighbor has 2 more than 3 times as many. Both together have 30 sheep. How many has each?

We will let x represent the number of sheep which the first farmer has. Then will $3x + 2$ represent the number which the second farmer has, because he has 3 times as many and 2 more. And since both together had 30, x and $3x + 2$ must together equal 30. Therefore, we have the equation

$$x + 3x + 2 = 30.$$

Transposing, $x + 3x = 30 - 2.$

Collecting $4x = 28.$

Dividing by 4, we find $x = 7,$

which is therefore the number the first farmer had;

and $3x + 2 = 23,$

which is therefore the number the second had.

Proof. $7 + 21 + 2 = 30.$

PROB. 3. A is 4 years older than B. If he were 10 years older than he is, he would be twice as old as B. What are their ages?

STATEMENT. Let $x =$ (that is, represent) the number of years in B's age; then will $x + 4 =$ the number of years in A's age, because he is 4 years older than B. And since if he were 10 years older than he is he would be twice as old as B, $x + 4$ and 10 must together equal $2x$. Therefore, we have the equation

$$x + 4 + 10 = 2x.$$

SOLUTION. Transposing, $x - 2x = -4 - 10.$

Collecting, $-x = -14.$

Changing signs, $x = 14$, No. of yrs. A's age,

and $x + 4 = 18$, No. of yrs. B's age.

Proof. $18 + 10 = 28.$

35. We see that there are three parts to the solution of each of the foregoing problems.

First. The representation of the unknown numbers by a letter, and by algebraic expressions containing that letter.

Second. The expression, in the form of an equation, of the equality of certain of these, the equality being either distinctly stated or indirectly implied in the problem.

Third. The solution of the equation to find the value of the unknown quantity.

The first two operations—the representation of the unknown quantities by letters and algebraic expressions, and the formation of equations from their known relations—are included in the one term, **Statement of the Problem.** The third operation is called the **Solution.** The Statement of the Problem is usually much the more difficult part, and should receive the close attention of the student.

The student has had practice in solving equations, and, in the last section, in forming equations from known relations between algebraic expressions. In the following exercises practice is given in the representation of unknown quantities by appropriate algebraic expressions. The student, having thus had experience in each of the three operations used in solving problems, will be able to solve readily those which are to follow.

MENTAL EXERCISE.

1. If x represent a certain number, what expression will represent 3 times that number?

2. What expression will represent 2 more than 3 times that number?

3. What expression will represent 5 less than 4 times that number?

4. If A has x number of cows, and B lacks 4 of having twice as many, and C has 10 more than B, what expressions will represent B's and C's number of cows?

5. If x represent a certain whole number, what expressions will represent the next two consecutive numbers?

6. If John be x years old, what expression will represent the number of years in his age 5 years ago?

7. What will represent his age 3 years hence?

8. If x represent one part of 100, what expression will represent the other part?

9. If the head of a certain fish be x inches in length, and its body be 3 inches longer than twice its head, and its tail be 2 inches shorter than its head, what expressions represent the lengths of its body and tail?

10. If x represent one number, and a second be 2 more than 3 times the first, and a third equal to the sum of the other two, what expressions will represent the last two numbers?

11. If a man walk 4 miles an hour for x hours, and then 3 miles an hour for $x + 4$ hours, what expressions will represent the number of miles travelled in each of the times?

SOLUTION OF PROBLEMS.

36. ***Rule.***—*I. Let x represent one of the unknown numbers, and find expressions for the others.*

II. Examine the given relations stated in the problem between the numbers which these expressions represent, and form an equation.

III. Solve this equation to find the values of x, from which the values of the other expressions can be found.

REMARK.—In such problems as are here given, x always represents some *number*. Do not say, Let x = John's age, or James's money, or the farmer's sheep; but say, Let x = the *number* of years in John's age, or the *number* of cents James has, or the *number* of the farmer's sheep. So the other expressions represent some *number* of something, and should be so stated.

PROBLEM. A farmer has 70 sheep divided between 2 fields. The number in the larger field is 4 times that in the smaller. How many are in each?

NOTE.—In the following statement the parts in *italics* need not be written down in performing the work, but should be given in the oral explanation of it.

STATEMENT. Let x = the number of sheep in the smaller field,

and $4x$ = the number in the larger field—*because there are four times as many as in the smaller field;*

then $x + 4x$ = the number in both fields.

But the problem states that there were in both 70 sheep. Therefore we have the equation,

$$x + 4x = 70.$$

The *solution* can now be easily made.

37. REMARK.—The student must be careful to find expressions for all the unknown quantities needed before trying to form the equation. The neglect of this is the cause of a large part of the difficulty often found in solving problems. Do not attempt to form the equation as soon as "Let x = the number," etc., has been written.

Let the student also be careful to give neat explanations. Simply reading off the work is not explaining it at all. Let a uniform plan of explanation be adopted. The one followed in this book is believed to be a very good one.

38. In explaining, *state what x, and other expressions, represent, giving the reason for each* after *the expression is stated. But in forming the equation, give the reasons for it* before *the equation is announced; so that it may be ready for its solution as soon as it is stated. Always introduce the equation with some* *such remark as, Hence we have the equation, that it may be at once distinguished from the expressions.*

PROBLEMS.

(To be solved in writing.)

1. Two sticks of wood measure together 18 inches, and one is 3 inches more than twice as long as the other. What is the length of each?

STATEMENT.—Let x = number of inches in shorter piece. Find expressions for inches of longer piece and for their united length.

2. A boy is reading a book of 216 pages. The number of pages that he has read is 21 less than twice the number that is left. How many pages has he read?

NOTE.—It is not necessary to let x represent the number called for in the answer. It should represent that unknown number which will give the simplest equation.

STATEMENT.—Let x = number of pages remaining. Find expressions for pages read and for whole number of pages.

3. At a certain election there were 360 votes cast, the successful candidate receiving a majority of 108 votes. How many did each receive?

4. At a certain election there were 148 votes cast. The successful candidate received 2 less than twice the number that the other received. What were the votes cast for each?

STATEMENT.—Find expressions for number of votes for unsuccessful candidate, for successful candidate, and for both together.

5. A merchant bought 10 bushels of corn and 6 bushels of wheat for $8.80, the wheat costing twice as much as the corn. What was the price of each?

STATEMENT.—Find expressions for number of cents per bushel for corn, and for wheat, for cost of corn, and for cost of wheat, and for cost of both together.

6. A and B start in business at the same time, A having 3 times as many dollars as B. A loses $300, and B doubles his money. They then have the same amount. How much had each at the first?

STATEMENT.—Find expressions for number of dollars A had at first, for number B had at first, for number A had after loss, for number B had after gain.

7. A tree 95 feet high was broken near the middle by a storm. The upper part hung down so that the top of the tree came within 15 feet of the ground. How much was left standing?

8. James has 6 more chestnuts than William; but James eats 10 while William finds 14. They now have together 44 nuts. How many has each at the last?

9. A farmer plants an acre each in corn, wheat, and oats. The oats yield 10 bushels more than the wheat, and the corn twice as much as the wheat. He gathered in all 130 bushels of grain. How many bushels were there of each?

STATEMENT.—Find expressions for number of bushels of wheat, of oats, of corn, and of all together.

10. A father lacks 2 years of being 3 times as old as his son, and their ages together amount to fifty years. What are their ages?

11. Divide the number 48 into two such parts that the greater shall be 4 less than 3 times the less part.

12. James and Henry go fishing. Henry pulls out of the water 3 times as many fish as James, but loses 12, while James only loses 2. They bring home 22 fish. How many belong to each?

13. A has twice as many dollars as B; but A loses $500, while B gains $300. They then have together $1300. How much had each at first?

14. The Leaning Tower of Pisa is 178 feet high, which is 9 feet more than 13 times the distance it leans over. How far does it lean?

15. The planet Jupiter has two more moons than the planet Mars is known to have, and Saturn has two more than as many as the other two together. In all they have 14 moons. How many moons are known to belong to each?

SECTION XIV.

REVIEW AND SUMMARY.*

NOTE.—The Articles at the end of the questions show where the answers may be found.

1. IN the expression $4a^2$, what is the 4 called? (Art. 4.)

2. What does it show? (Art. 4.)

3. Express $4a^2$ without any coefficient. (Art. 4.)

4. What is the small 2 over the right of the a called (Art. 13.)

5. What does it indicate? (Art. 13.)

6. Express a^2 without any exponent.

7. Express $4a^2$ without either coefficient or exponent.

8. Express a^3x^2 without any exponents.

9. What are the parts of the expression $4a - 3b + x$ called? (Art. 15.)

10. What is a Polynomial? (Art. 17.)

11. What is a polynomial of two terms called? (Art. 18.)

12. What is a polynomial of three terms called? (Art. 19.)

13. What do we call such an expression as $7ax$? (Art. 16.)

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* The pupils are expected to recite this review with the book open before them. The teacher may call out the questions by number, or may read them. If the class appear to be deficient in any subject, let them go back to the Articles treating of that subject, and be reviewed and drilled on them. The class should not pass on till everything is fully comprehended.

14. Give the name of each of the following expressions:

$$4ab + c^2,$$
$$5x + 2y - 3z,$$
$$2a^3x^2y^4.$$

15. Collect $4a + 3a - 2a$.

16. Can we collect $a + b$? Give reason for answer.

17. Can we collect $a^2 + a$? Give reason for answer.

18. Collect $2a^2 + ab - a^2$.

19. Why does not $a^2b + a$ equal a^3b?

20. In the monomial $6x^4$, what sign may be understood between the 6 and x? (Art. 4.)

21. What are meant by *positive* terms and *negative* terms?

22. In the polynomial $5y^2 + 4yz - 3z^2$, which terms are positive and which are negative? (Art. 21.)

23. What is meant by similar quantities? (Art. 14.)

24. Why can similar terms only be collected? (Art. 14.)

25. Collect $3x - 8x$; explaining the process. (Art. 20.)

26. State the difference between Addition and Collecting. (Art. 23.)

27. In solving the equation $5x - 2x = 17 + 1$, what two operations are performed? (Art. 10.)

28. In solving the equation $2x + 4 = 12 - 6x$, what *three* operations are performed? (Art. 32.)

29. State the rule for addition in Algebra. (Art. 24.)

30. What does $-7ab + 7ab$ equal?

31. What does $0 + 8ab$ equal?

32. What does $4c^3 - 4c^3$ equal?

33. What does $0 - 6c^3$ equal?

34. What two quantities added to $4x^2$ will give $-6x^2$? (Art. 26.)

35. To $-7ay$ will give $3ay$?

36. What principle always applies to examples like the last two?

37. What is Subtraction? (Art. 28.)

38. What is the Remainder?

39. How is it found? Rule (Art. 28).

40. Subtract $\begin{array}{r} 3a \\ \underline{b} \end{array}$

41. Subtract $\begin{array}{r} 4a^2x \\ \underline{2a^2x} \end{array}$

42. What are the members of an equation? (Art. 8.)

43. When a quantity is taken away from one side of an equation, what must be done to the other side? and why? (Art. 31.)

44. What is meant by Transposition?

45. What is the object of transposing some of the terms of the equation $5x - 4 = 2x + 5$?

46. Why are the signs of terms changed when they are transposed? (Art. 31.)

47. What is meant by the *statement of a problem?*

48. Why is it incorrect to write, Let $x =$ A's money?

49. What does x always stand for? (Art. 37.)

50. Give the two meanings of the symbol 0.

51. If $-x$ equals -4, what does $+x$, or x, equal?

52. If we subtract 10 from 7, why is the result -3?

53. If a thermometer indicates 12°, and the mercury falls 20 degrees, what will it indicate?

54. What do we mean by saying the thermometer indicates $-5°$?

Transpose the terms of the following equations:

55. $13 = 5x - 4x - 8 + 6x$.

56. $4x - 10 - 5x + 6 + 2x = 0$.

SECTION XV.

MULTIPLICATION BY MONOMIALS.

MULTIPLICATION BY NUMBERS.

39. Ex. Multiply $4ax$ by 5.

Solution. Since $4ax$ means 4 times ax, 5 times that is $20ax$.

In the same way, $8x^2y \times 2 = 16x^2y$.

Perform the multiplication indicated in the following

MENTAL EXERCISES.

1. $7ay^2 \times 3$.
2. $-5abc \times 6$..
3. $3a \times 3$.
4. $9bc^2y \times 4$.
5. $-8x^5 \times 3$.
6. $6xy^3 \times 8$.

Where there are several connected quantities to be multiplied by the same number, we multiply each separately.

7. $$\begin{array}{l} 4ax + 2x^2 - 3xy \\ \underline{6} \end{array}$$

8. $$\begin{array}{l} 7b^2 - 4bc + 8c^2 \\ \underline{3} \end{array}$$

9. $$\begin{array}{l} 5axy + 6x^3y - 8y^2 \\ \underline{5} \end{array}$$

10. $$\begin{array}{l} -2acd + 5c^2 - 10a^3 \\ \underline{8} \end{array}$$

40. Multiply 6 by -3.

Explanation. The sign $-$ denotes subtraction. When written before a quantity, it usually means that the quantity is to be subtracted from *something*. But before the multiplier 3 it means, not that 3 is to be subtracted, but that 3 times the multiplicand 6 is to be subtracted; that is, that 18 is to be subtracted. Now, in subtraction we found that we must change the sign of the subtrahend, or quantity to be subtracted. Hence, since 18 is to be subtracted, we change its sign, and have for the product -18. That is,

$$6 \times -3 = -18.$$

Multiply -7 by -4.

EXPLANATION. The sign $-$ before the multiplier 4 means that 4 times the multiplicand is to be subtracted from something. 4 times -7 gives -28. If this is to be subtracted, its sign, as before, must be changed; that is, the product is $+28$. Therefore, $-7 \times -4 = 28$.

We have, therefore, the following

Rule.—In multiplying by a negative quantity, change the $-$ signs to $+$, and $+$ signs to $-$.

ILLUSTRATION.

$3x^2y \times -7 = -21x^2y$; $-8ab^3d \times -4 = 32ab^3d$.

Perform the multiplication indicated in the following

MENTAL EXERCISES.

1. $4m^2n \times 6$.
2. $9b^3c^2 \times -3$.
3. $11xy^2 \times -5$.
4. $$\begin{array}{l} 4ab + 3b^2 - 2c \\ \underline{-3 \qquad\qquad\quad} \end{array}$$
5. $$\begin{array}{l} 8m^2 - 3mn + n^2 \\ \underline{-4 \qquad\qquad\quad} \end{array}$$
6. $$\begin{array}{l} 9ac - 4cd + 6ab \\ \underline{5 \qquad\qquad\qquad} \end{array}$$
7. $$\begin{array}{l} 7x^2 - 5xy - 4y^2 \\ \underline{-7 \qquad\qquad\quad} \end{array}$$
8. $$\begin{array}{l} -11b^3c + 4b^2c^2 - 5bc^3 \\ \underline{\qquad 6 \qquad\qquad\qquad} \end{array}$$

MULTIPLICATION BY LETTERS.

41. Let us multiply a by b.

Since writing letters together expresses their product, $a \times b = ab$. Also, $ab \times c = abc$.

What do the following expressions equal?

NOTE.—Give the letters in their alphabetical order.

1. $ac \times b$.
2. $3\,acx \times 2\,by$.
3. $7\,mn \times 2\,ax$.
4. $3\,dm \times 7\,acx$.
5. $4\,x^2 \times 2\,y^2$.
6. $5\,a^3 \times 4\,c$.

42. It is plain from these examples that the letters of the product are those of the multiplier and multiplicand. Now, multiply a^3 by a^2. The exponent 3 in the multiplicand shows how many times the letter is meant to be taken as a factor. So the 2 in the multiplier shows how many times it is taken there. In the product, therefore, it must be taken as many times as both these together, or 5 times; that is, $a^3 \times a^2 = a^5$. Hence the

Rule.*—Where the same letter occurs in both the multiplier and multiplicand, add together its exponents in these to find its exponent in the product.*

ILLUSTRATION. $b^2 \times b^5 = b^7$, and $3\,ax^3 \times -2\,a^2x^2 = -6\,a^3x^5$.

Perform the multiplication indicated in the following

MENTAL EXERCISES.

1. $b^4 \times b^3$.
2. $c^2 \times c^4$.
3. $m^3 \times m^5$.
4. $ab \times b^3$.
5. $a^2m \times am^2$.
6. $4\,c^3y^2 \times 3\,c^2y^3$.
7. $8\,a^3 \times 4\,a^4$.
8. $-7\,x^2y^2 \times 3\,y^3$.
9. $-11\,m^3x \times 4\,mn^2$.
10. $4\,a^2cx \times 6\,ac^3x^5$.
11. $5\,b^2c^4y \times 7\,ab^3cy^2$.
12. $9\,c^2x^3y^4 \times -abc$.
13. $7\,c^3d^2 \times -3\,ad^4x$.
14. $-12\,m^2x^4 \times 5\,m^6y^2$.
15. $20\,x^6y^3 \times 4\,a^3xy^3$.
16. $-6\,b^2c^3d \times -8\,a^4b^2$.

17. $$\begin{array}{l} 5\,b^3c - 4\,b^2 + 2\,c^5 \\ \underline{6\,b^2c^3 \qquad\qquad\quad} \end{array}$$

18. $$\begin{array}{r} 7\,ax^3 + 9\,a^2xy^2 + a^4y^3 \\ \underline{-5\,a^3y^2 \qquad\qquad\qquad\qquad} \end{array}$$

19. $$\begin{array}{l} 14a^3mn^2 - 12am^3n + 4m^5 \\ \underline{2mn^4 \qquad\qquad\qquad\qquad} \end{array}$$

20. $$\begin{array}{l} -6c^3d^2 - 5c^2d^3 + 8cd^4 \\ \underline{-7c^3d^4 \qquad\qquad\qquad} \end{array}$$

21. $$\begin{array}{l} 8x^3y + 4xy^3 - 5y^2 \\ \underline{9xy^2 \qquad\qquad\quad} \end{array}$$

22. $$\begin{array}{l} \quad 10m^2x - 5mx^2 - my \\ \underline{- \quad 8x^2 \qquad\qquad\qquad} \end{array}$$

23. $$\begin{array}{l} \quad 7y^3 - 4y^2 + 8y \\ \underline{-3xy^2 \qquad\qquad} \end{array}$$

24. $$\begin{array}{l} 9a^3b - 5a^2x + 3axy \\ \underline{6a^2y \qquad\qquad\qquad} \end{array}$$

25. $$\begin{array}{l} 6ac^2x - 5a^2xy + 13ay^4 \\ \underline{\ ac^2 \qquad\qquad\qquad\qquad} \end{array}$$

26. $$\begin{array}{l} -13bm^3 + 8b^3m - 5b^5 \\ \underline{\quad -bm^2 \qquad\qquad\qquad} \end{array}$$

27. $$\begin{array}{l} \quad axy + xyz - yz \\ \underline{12a^2z \qquad\qquad\quad} \end{array}$$

28. $$\begin{array}{l} 5y^3 + 5x^3 - 4xz \\ \underline{3xy \qquad\qquad\qquad} \end{array}$$

29. $$\begin{array}{l} 14c^2d^3 - 18cd^2 + 2x^3 \\ \underline{\ 2d^2y \qquad\qquad\qquad\quad} \end{array}$$

30. $$\begin{array}{l} \quad 15m^2x^3 - 12mx^4 + xy^2 \\ \underline{- \quad 3a^3x^2 \qquad\qquad\qquad} \end{array}$$

SECTION XVI.

REVIEW OF ADDITION.

43. Only *similar terms*—that is, terms whose letters and exponents are exactly alike—can be collected into one term.

Rule.*—To find the sum of the similar terms, add together the coefficients of all the positive terms, and also of all the negative terms. Take the difference between these sums and give it the sign of the greater.*

It is not necessary that the similar terms should all be in the same column, though it is convenient to have them so when possible.

EXAMPLES.

1. $$\begin{array}{r} 4ay + 6y^2 + 2z \\ -3ay - 4y^2 + 5z \\ ay + 2y^2 - 8z \\ \hline \end{array}$$

2. $$\begin{array}{r} 2c^3d - 7cd^2 + 4d^3x \\ -6c^3d + 5cd^2 \qquad\quad \\ 3c^3d \qquad\qquad - 8d^3x \\ \hline \end{array}$$

3. $$\begin{array}{r} 5a + 4x \\ 3x - 2a \\ \hline \end{array}$$

4. $$\begin{array}{r} 3ay - 9y^3 \\ 5y^3 + 4ay \\ \hline \end{array}$$

5. $$\begin{array}{r} 8x^2y - 3xy^2 \\ 5xy^2 - 3x^2y \\ \hline \end{array}$$

6. $$\begin{array}{r} a^2 + ax \qquad \\ ax + x^2 \\ \hline \end{array}$$

7. $$\begin{array}{r} m^2 - mn \qquad \\ mn - n^2 \\ \hline \end{array}$$

8. $$\begin{array}{r} 4c^2 - 2cx \qquad\quad \\ 2cx - x^2 \\ \hline \end{array}$$

9. $$\begin{array}{r} a^2 - ab + ac \qquad\qquad\quad \\ ab - b^2 - bc \qquad \\ -ac + bc - c^2 \\ \hline \end{array}$$

10. $$\begin{array}{r} x^4 - x^3y + x^2y^2 \qquad\qquad\quad \\ x^3y - x^2y^2 + xy^3 \qquad \\ x^2y^2 - xy^3 + y^4 \\ \hline \end{array}$$

11. $$\begin{array}{r} 8a^5 + 12a^4x - 8a^3x^2 \qquad\qquad\qquad \\ -\quad 4a^4x - 6a^3x^2 + 4a^2x^3 \qquad \\ 2a^3x^2 + 3a^2x^3 - 2ax^4 \\ \hline \end{array}$$

12. $$\begin{array}{r} 4a^2 - 3x^2 + 4ax \\ 3ax - 2x^2 + 3a^2 \\ 4x^2 - 5a^2 - 2ax \\ \hline \end{array}$$

13. $$\begin{array}{r} 30c^2 - 18cd \qquad\qquad \\ 35cd - 21d^2 \\ \hline \end{array}$$

14. $$\begin{array}{r} 2a^2 - 3ab \qquad\quad \\ 2ab - 3b^2 \\ \hline \end{array}$$

15. $$\begin{array}{r} x^6 + x^4y^2 + x^2y^4 \qquad\qquad\qquad \\ -x^5y - x^3y^3 - xy^5 \qquad \\ -x^4y^2 - x^2y^4 - y^6 \\ \hline \end{array}$$

SECTION XVII.

MULTIPLICATION BY A POLYNOMIAL.

44. Ex. Multiply $2a + 3b$ by $a + b$.

SOLUTION. $a + b$ times $2a + 3b$ equals a times $2a + 3b$ plus b times $2a + 3b$. We therefore multiply first by a, and then by b, and add together the two products. The operation is performed thus:

$$\begin{array}{ll}
\text{Multiplicand,} & 2a + 3b \\
\text{Multiplier,} & \underline{\;\,a + \;\;b\qquad\quad} \\
 & 2a^2 + 3ab \\
 & \underline{\qquad\quad 2ab + 3b^2} \\
\text{Product,} & 2a^2 + 5ab + 3b^2
\end{array}$$

EXPLANATION. Multiplying first by a, and beginning at the left, we obtain as the first partial product $2a^2 + 3ab$. Multiplying then by b, we have as the second partial product $2ab + 3b^2$. Adding these together, we obtain the total product, $2a^2 + 5ab + 3b^2$.

Ex. Multiply $4c^2 - 3cx + x^2$ by $2c - x$.

Here the partial products are obtained by multiplying by $2c$ and $-x$. The operation is as follows:

$$\begin{array}{rrrr}
4c^2 & -\;3cx & +\;x^2 & \\
\underline{2c} & \underline{-\;\;\;x} & & \\
8c^3 & -\;6c^2x & +\;2cx^2 & \\
 & \underline{-\;4c^2x} & \underline{+\;3cx^2} & \underline{-\;x^3} \\
8c^3 & -\;10c^2x & +\;5cx^2 & -\;x^3
\end{array}$$

The explanation is the same as for the former example. In explaining the second multiplication the pupil must be careful to say, Multiplying by $-x$, and not simply by x. He must also remember, when multiplying by a negative term, to change all the signs in the partial product. The final operation is like all other addition.

In the same way perform the following

WRITTEN EXAMPLES.

1. Multiply $a + x$ by $a + x$.
2. Multiply $a - b$ by $a - b$.
3. Multiply $2a - 3b$ by $a - b$.
4. Multiply $5c - 3d$ by $6c + 7d$.
5. Multiply $a - m$ by $a + m$.
6. Multiply $3c + 4d$ by $3c - 4d$.
7. Multiply $a + b + c$ by $a + b$.
8. Multiply $a - b - c$ by $a - c$.
9. Multiply $5a^2 + 3ax + x^2$ by $2a + 5x$.
10. Multiply $a^2 + ax + x^2$ by $a - x$.
11. Multiply $3y^2 - 2yz + 4z^2$ by $y - 3z$.
12. Multiply $a + b$ by $c - d$.
13. Multiply $7x^2 - 9xy - 4y^2$ by $3x - 2y$.
14. Multiply $x^2 - xy + y^2$ by $2x + 2y$.
15. Multiply $x^2 - 2xy + y^2$ by $x - y$.
16. Multiply $x^2 + y^2$ by $x^2 - xy + y^2$.
17. Multiply $2a + 3b - c$ by $2a - 3b + c$.
18. Multiply $6a^2x + 3ax^2 - x^3$ by $a^2 + 2x^2$.
19. Multiply $9a^2 - 4c^2$ by $9a^2 + 4c^2$.
20. Multiply $5a^3 - 4a^2$ by $3a^2 + 2a$.
21. Multiply $c^3 + 3c^2m - 3cm^2 + m^3$ by $2c + m$.
22. Multiply $a^3 - a^2b + ab^2 - b^3$ by $a + b$.
23. Multiply $a^2 + 2ax + 3xy$ by $a - 2x + y$.
24. Multiply $5x^3 - 3x^2 + 4$ by $6x^2 - 3$.
25. Multiply $a + b - c$ by $a + x - y$.
26. Multiply $3x^4 - 2x^3 + 4x^2 - 6x - 2$ by $x^2 - 2x + 1$.

27. Multiply together $a + b$, $a - b$, and $a^2 + b^2$.

28. Multiply together $a^2 + ax + x^2$, $a - x$, and $a + x$.

29. Multiply together $2c - 1$, $3c + 2$, and $c^2 - 4c + 4$.

30. Multiply together $ax - x^2$, $3ax^2$, and $a + x$.

SECTION XVIII.

USE OF THE PARENTHESIS.

45. Thus far we have used signs to indicate the operations between single terms. It is often necessary to indicate operations between polynomials; as, the addition of $a + b$ to $a - b$. This is done by placing each quantity in a parenthesis, and writing the sign + between the parentheses, thus $(a+b)+(a-b)$. The parentheses show that the quantities, as wholes, are to be added together.

In the same way we can express the subtraction of $a - 2x$ from $3a + x$.

$$(3a + x) - (a - 2x).$$

This is the same as writing—

$$\begin{array}{ll} \text{Minuend,} & 3a + x \\ \text{Subtrahend,} & \underline{a - 2x} \end{array}$$

In the same way the multiplication of polynomials may be shown. $(5a + x - y) \times 3x$ means that the whole quantity in the parenthesis is to be multiplied by $3x$. $(b + c) \times (b - c)$ means that $b + c$ is to be multiplied by $b - c$. When the quantities are to be multiplied it is usual to omit the sign ×. The multiplication is indicated by simply writing together the quantities, as with letters. Thus, $3(a + b)$ means 3 times $a + b$. It equals $3a + 3b$.

46. Indicate the subtraction of $c - d$ from $a + b$.

Solution. Putting each quantity in a parenthesis, and connecting them by a minus sign, we have

$$\overset{\text{MIN.}}{(a + b)} - \overset{\text{SUB.}}{(c - d)}.$$

If we wish to find the value of the expression, we proceed exactly as though it were written

Minuend, $a + b$
Subtrahend, $\underline{c - d}$

That is, we *change the signs of the quantity to be subtracted, and add it to the other.* Therefore $(a + b) - (c - d)$ will equal $a + b - c + d$.

If any of the terms were alike, we could collect them.

Thus, $(2c + 3d) - (c - d) = 2c + 3d - c + d$,
and $2c + 3d - c + d = c + 4d$.

The first operation here is called *removing the parenthesis.*

The first parenthesis in the last two cases is not really necessary. The sign $-$ indicates that the second quantity is to be subtracted from whatever it is connected with. The subtraction in each case would have been indicated by writing $a + b - (c - d)$ and $2c + 3d - (c - d)$.

Remove the parenthesis from the expression—

Ex. $c + d - (x + y)$.

Solution. In performing the indicated subtraction we remove the parenthesis and change the signs of the terms in it, obtaining $c + d - x - y$.

Note.—The explanation need not be given unless the teacher calls for it.

Remove the parentheses from the following expressions:

MENTAL EXERCISE.

1. $3m - (2n + x)$.
2. $4a + b - (-x + y)$.
3. $6y - (a + b)$.
4. $2am - (3mn - 2m^2)$.
5. $4ay - (3ax + x^2)$.
6. $9mx - (2m^2 - x^2)$.
7. $3bc - (2by + cx) - 3ab$.
8. $2y^2 - (3ay + 2xy) - (4yz - z^2)$.
9. $4ax + (2x - y) - (3x^2 - y^2)$.

47. Remove the parenthesis from $2a - (a - x)$ and collect the similar terms.

Solution. Performing the indicated subtraction, we obtain

$$2a - a + x.$$

Collecting, we have $\qquad a + x.$

It is usual to conceive the signs to be changed, and collect directly from the first expression.

Remove the parentheses from the following expressions and collect the similar terms:

MENTAL EXERCISE.

1. $4x - (-2x - y)$. 5. $5c - 2d - (8c + 7d)$.
2. $5y^2 - (y^2 + z)$. 6. $5a^2 + 4a - (2a^2 + 2a)$.
3. $3a + 2b - (2a + b)$. 7. $7a^2x - (-2a^2x - 3a^2)$.
4. $7x - y - (3x - 4y)$. 8. $8xy - 3y^2 - (4y^2 + 3xy)$.

48. Perform the multiplication indicated in the expression

$$3(a + 2b).$$

Solution. We know that the operation of multiplication is here indicated, because the quantities 3 and $(a + 2b)$ are written together without any sign between them. Performing the multiplication, we have $3a + 6b$.

Perform the multiplication indicated in the following

MENTAL EXERCISE.

1. $7(2x - 3y + z)$. 5. $-4ay(a^2 - 2ay)$.
2. $-4(3x - 2y)$. 6. $(4mx - 2x^2)3x$.
3. $2a(a - b)$. 7. $(3a^2x - x^3)x \times -2ax$.
4. $7bc(2b^2 - 3c^2)$. 8. $-(4xy - y^2)xy^3$.

49. Remove the parentheses from the equation

$$5(2x - 1) = 10x - (x + 3).$$

Solution. Performing the indicated multiplication and subtraction, we have

$$10x - 5 = 10x - x - 3;$$

50. Remove the parentheses from the following equations by performing all indicated operations of addition, subtraction, or multiplication:

MENTAL EXERCISE.

1. $5 - (x + 3) = 7 + 2(x - 1)$.
2. $x + 2(3 - x) = 4 - (2 + x)$.
3. $y - (y - 4) = 4(y + 6)$.
4. $5(6 - x) = x - 2(x - 3)$.
5. $15 - 4(7 + x) = 9(2x - 3)$.
6. $-4(6 - 2x) = 12 - 4(x - 3)$.

51. Given $5 - (3 + x) = 2(x + 1) - 9$, to find the value of x.

Solution. Performing indicated operations to remove parentheses, we have $5 - 3 - x = 2x + 2 - 9$.

Transposing, $-x - 2x = -5 + 3 + 2 - 9$.

Collecting, $-3x = -9$.

Dividing by 3, we find $-x = -3$.

Hence, $x = 3$.

Proof. Substituting the value of x in the first equation,

we have $5 - (3 + 3) = 2(3 + 1) - 9$.

Reducing, $5 - 6 = 8 - 9$.

Collecting, $-1 = -1$.

When explaining, be careful always to mention the four processes—performing indicated operations, transposing, collecting, and dividing by the coefficient of the unknown quantity.

Solve the following equations:

WRITTEN EXAMPLES.

As a Mental Exercise, let the student remove the parentheses.

1. $3 - (1 - x) = 5$.
2. $2x - 4 = 3(x - 3)$.
3. $x - 4(7 - 2x) = 2(2x + 6)$.

4. $4 + 6(3x - 8) = 14 - 5(x - 16)$.
5. $3(2x + 6) = 5(x - 1) + 27$.
6. $x + 4 - (3 - 2x) = 2x + 9$.
7. $3(x - 12) = 2(32 + 2x) - 87$.
8. $(2x + 3) - (x + 1) = 10 - 2(x - 2)$.
9. $5 + 4(x + 6) = -(2x - 40) + 31$.
10. $2x - 10 - (x - 4) = 27 - (2x + 6)$.
11. $5x - (6x - 7) = 3(2x + 4) - 8x$.
12. $4(6x - 3) = 9x - (x - 4)$.
13. $2x + 2(3x + 2) = 60 - (x + 2)$.

52. An expression like $(a + b)(a - b)$ indicates that the two quantities in the parentheses are to be multiplied together, because they are written together, without any sign between them.

Perform the multiplication indicated in the following. Multiply as in Art. 44.

WRITTEN EXAMPLES.

1. $(2 - x)(4 + x)$.
2. $(x + 3)(x - 7)$.
3. $(2x + 8)(5x - 3)$.
4. $(a + 2b)(a - 3b)$.
5. $(x + 6)(2x - 1)$.
6. $(3x - 5)(4x - 2)$.

53. Given $(x - 3)(x + 1) = (x - 1)(x - 2)$, to find the value of x.

Solution. Performing the indicated multiplication, we obtain

$$x^2 - 2x - 3 = x^2 - 3x + 2.$$

Since x^2 occurs in both members of the equation, we may subtract it from both without destroying the equality. Doing this, we obtain

$$-2x - 3 = -3x + 2.$$

Transposing, we have $3x - 2x = 3 + 2$.

Collecting, we find $x = 5$.

Proof. Substituting the value of x in the first equation, we have

$$25 - 10 - 3 = 25 - 15 + 2.$$

Collecting, $25 - 13 = 25 - 13.$

Collecting again, $12 = 12.$

Given $(x + 3)(x - 5) = 6 - (5 - x)(2 + x)$, to find the value of x.

SOLUTION. The second member of this equation means 6 minus the product of $5 - x$ and $2 + x$. We will therefore multiply $5 - x$ by $2 + x$, and write the product after the 6, with all of its signs changed, since it is to be subtracted. Hence we say, performing the indicated multiplications and subtraction, we have

$$x^2 - 2x - 15 = 6 - 10 - 3x + x^2.$$

Striking out x^2 in each member and transposing, we have

$$-2x + 3x = 15 + 6 - 10.$$

Collecting, we find $x = 11.$

Proof. Substituting the value of x in this first equation, we have

$$121 - 22 - 15 = 6 - 10 - 33 + 121.$$

Collecting, $121 - 37 = -37 + 121.$

Solve the following equations:

WRITTEN EXAMPLES.

1. $(x - 3)(x + 2) = x^2 - 13.$
2. $x(x + 3) = x^2 + 12.$
3. $(x + 1)(x - 8) = (x + 2)(x + 4) - 29.$
4. $(4x - 2)(x + 1) = (2x + 5)(2x - 3) + 2x + 1.$
5. $2x(3x + 2) = 6x^2 + 3x + 5.$
6. $x - 5 = 6(2x + 2) - 28.$
7. $(2x - 5)(2x - 7) = 6 + 7x - 2(14 + 6x - 2x^2).$
8. $(3x - 1)(3x + 1) = 4x + (3x - 1)(3x - 1) + 4.$

9. If 4 times $x - 2$ be equal to twice $x + 1$, what is the value of x?

10. If 3 times $x + 3$ be 2 more than 5 times $x - 1$, what is the value of x?

11. If 3 times $5x + 2$ be 5 less than 7 times $x + 5$, what is the value of x?

12. If the difference between twice a number and 50 be equal to the sum of that number and 2, what is the number?

SUGGESTION.—Let $x =$ the number. In this and the succeeding examples form the equation without finding any intermediate expressions. Where the difference of two quantities is called for, place before the sign $-$ the quantity mentioned first.

13. If the difference between 3 times a number and 9 be equal to the difference between 7 and the number, what is the number?

14. If 3 times the difference between a certain number and 7 be equal to 1 more than the number, what is the number?

15. If 3 more than twice the difference between 10 and a certain number be the same as 3 times the number increased by 1, what is the number?

16. If 5 less than 7 times the difference between a number and 3 be equal to 10 less than 3 times the number, what is the number?

17. If 3 times the sum of 3 times a certain number and 2 be equal to 12 more than 8 times that number, what is the number?

18. If 2 less than 3 times a certain number be 3 more than 4 times the difference between that number and 3, what is the number?

19. If 5 times the sum of a number and 7 be 4 less than 6 times the sum of twice the number and 8, what is the number?

SECTION XIX.

SOLUTION OF PROBLEMS.

MENTAL EXERCISE IN THE CONSTRUCTION OF EXPRESSIONS.

1. If $3x$ represents a person's age, what expression represents his age 5 years ago? What twice his age 5 years ago?

2. If B starts from the same place 6 hours after A, and overtakes him in x hours, what expression represents the hours A travels?

3. What expression represents the number of miles A travels if he goes 4 miles an hour?

4. If a boatman can row x miles an hour, what expression will represent the number of miles that he can row in 1 hour down a stream whose current is 2 miles an hour? What the number of miles in 3 hours?

5. What represents the number of miles he can row up stream in 4 hours?

6. A man works x days out of 30, and is idle the remainder. At 50 cents a day, what represents the board he must pay for his idle days?

7. If a man has a certain number of cows, and buys twice as many more, he will have how many times his first number?

8. If he has $x - 5$ cows, and buys 4 times as many more, what expression will represent the number he will then have?

9. If a rectangular tract of land is x yards wide, and 5 yards longer than wide, what represents its length, and what its contents?

10. If a tract of land is x yards wide and $x + 6$ yards long, and a second is 5 yards wider, but 2 yards shorter, what represents its dimensions and what its contents?

WRITTEN EXERCISES IN THE CONSTRUCTION OF EXPRESSIONS AND EQUATIONS.

1. A boy has x cents; what represents his number of cents after spending 10? What will represent his number of cents if he now earn as much more as he has? What will represent the number if he spend 50? If he then earn twice as much more, what will represent his amount? If he now has 10 cents less than twice as much as at first, what will be the equation? Solve it.

2. A merchant starts in business with x dollars. At the end of the first year he has trebled his money, and deducted $500 for expenses. What expression represents the number of dollars he then has for beginning business the second year? By the end of the second year he has *doubled* the money that he had at the beginning of that year, but has deducted again $500 for expenses. What expression represents the number of dollars that he now has? If this be 3 times as much as he had at first, what will be the equation? Solve it.

54. The student will bear in mind the instructions given before about the solution of problems, some of which are repeated:

Let x represent one of the unknown quantities, and find expressions for the others.

From the given relations between these quantities construct an equation. Solve the equation to find the value of x, and from that find the values of other needed expressions.

In explaining, *give the reasons for the expressions* after *the expressions are given. But give the reason for the equation* before *the equation is given.*

If much difficulty is found in constructing the equation, let the expressions for the various unknown quantities be very fully stated and their relations carefully examined.

The solution of the following problem illustrates these instructions:

PROB. A farmer takes a ton of hay to market, and with its proceeds buys 9 yards of cloth, and has \$2.75 remaining. Had the cloth been 25 cents per yard cheaper, he could have bought 10 yards and had \$4 remaining. What was the price per yard of the cloth?

NOTE.—In the following statement and solution the *reasons* are given in italics, and should be given in the oral explanation, but need not be written down when the example is worked on the slate or blackboard.

STATEMENT. Let $x =$ the number of cents per yard for the cloth;

then $9x =$ the cost, in cents, of 9 yards,

and $9x + 275 =$ the whole amount, in cents, received for the hay—*because the farmer bought 9 yards of cloth, and had \$2.75 remaining.*

$x - 25 =$ price, in cents, had the cloth been 25 cents less per yard;

$10(x - 25) =$ cost, in cents, of 10 yards at that price;

$10(x - 25) + 400 =$ the whole amount, in cents, received for the hay—*because, had the price been 25 cents less, he could have bought 10 yards, and had \$4.00 remaining.*

And since both $9x + 275$ *and* $10(x - 25) + 400$ *represent the amount received for the hay, they must be equal to each other.*

Hence, *we have the equation,*

$$10(x - 25) + 400 = 9x + 275;$$

SOLUTION. *Performing the indicated multiplication,*

$$10x - 250 + 400 = 9x + 275.$$

Transposing and collecting, we find

$x = 125$, the price of the cloth.

Proof. $9 \times 125 + 275 = 10 \times 100 + 400.$

Collecting, $1400 = 1400.$

PROBLEMS.

NOTE.—Suggestions on the solution of the problems on this page will be found on page 245.

1. A man's age is 3 times his son's age. 6 years ago it was 5 times his son's age. How many years old are each?

2. John is 6 years older than James. In 5 years 3 times John's age will be equal to 4 times James's. How many years old are each?

3. A man's boat broke loose and drifted down stream at the rate of 2 miles per hour. 6 hours afterward he secured another boat, and started after it at the rate of 5 miles per hour. In how many hours will he overtake it?

4. A man can row 3 miles an hour in still water. He rows with the current of the river, in 1 hour, a distance that it takes him 2 hours to row back. How many miles per hour does the current move?

5. A and B are walking together. After going 3 miles, A finds he has left his watch behind. If B goes on at the rate of 3 miles per hour, and A goes back after the watch and returns, travelling 4 miles per hour, in how many hours will he have overtaken B?

6. A man agreed to take $2 a day for every day he works, and pay $1.25 for board for every day he is idle. At the end of 24 days he receives $35. How many days has he worked?

7. A man agreed to take a certain sum for every day that he worked, and to pay $1.50 less than that sum for every day that he was idle. He worked 10 days out of 16, and was paid $25. What amounts were his wages and board?

8. A man hired 3 men, 5 women, and 8 children. The women received 3 times as much as the children, and the men received 35 cents a day more than the women. Their daily wages amounted to $10.65. How much did each receive for a day's work?

9. George is 15 and Thomas is 18. How many years ago was Thomas twice as old as George?

10. A man bought a horse and harness and carriage. The horse cost $100 more than the harness, and the carriage cost as much as the horse and harness together. The three together cost $100 more than twice the cost of the horse. How many dollars did each cost?

11. When a menagerie-keeper, who is 6 feet high, stands up on an elephant, he is just the height of a giraffe, which is 4 feet less than twice as high as the elephant. What are the heights of the giraffe and elephant?

12. A man has a certain number of sheep. He sells 10, buys as many more as he has, and sells 34. He then buys twice as many as he had left, when he has 2 less than at first. How many had he at first?

13. In a certain school there are 12 more girls than boys; but 10 more girls came and 4 boys left, when there were twice as many girls as boys. How many were there of each at first?

14. A gardener laid out a plot of ground in the form of a rectangle, 3 feet longer than it was wide. He laid out another rectangular plot 6 feet wider and 5 feet shorter, and found that the two were exactly the same size. What were the dimensions of each?

15. 5 yards of muslin could be bought for a certain sum; but the price having fallen 2 cents a yard, 6 yards can now be purchased for the same sum. What was the cost of a yard at first?

16. A merchant starts business with a certain sum of money. He doubles it the first year, but deducts $750 for expenses. In the second year he doubles the amount left at the end of the first, and deducts $1000 for expenses. During the

third year he again doubles the amount he had at the beginning of that year, and again deducts $1000 for expenses. He has now twice as much as he had at first. How many dollars had he at first?

SUGGESTION.—Find expressions for the number of dollars at the end of the first, the second, and the third years.

17. A merchant started in business with a certain amount of capital. The first year he doubled his money, but deducted $1500 for expenses. The second year he trebled his money, and again deducted $1500 for expenses. He now had nothing left. How much had he at first?

18. A person proposed to give 6 cents to each of the boys of a party and 5 to each of the girls, there being 32 more girls than boys. One more girl being added to the company, the same amount of money was just sufficient to give 5 cents to each person. How many boys and girls were there at first?

19. A man is 4 miles from home. How much further can he ride at the rate of a mile in 10 minutes if he has to walk back to his home at the rate of a mile in 15 minutes, so as to reach it in 2¼ hours.

SUGGESTION.—Find expressions for distance he would walk and time he would both ride and walk.

20. Four boys agree to contribute the same amount each for a sled, but a fifth boy joining them they decide to give each 20 cents more than they had intended and buy a better sled, costing $2 more than the other. How much did they pay for the sled?

SUGGESTION.—Let x = share of each boy at first.

SECTION XX.

EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

55. What is the value of x in the equation $5x + y = 22$ if $y = 2$?

Solution. Putting 2 in place of y, the equation becomes

$$5x + 2 = 22.$$

Transposing and collecting, $5x = 20.$

Dividing by 5, we find $x = 4.$

This operation of putting the value of a letter in place of the letter itself is called **Substitution.**

Ex. What is the value of y in the equation $3x + 4y = 46$ if $x = 6$?

Solution. Substituting the value of x, we have

$$18 + 4y = 46.$$

Transposing and collecting, $4y = 28.$

Dividing by 4, we find $y = 7.$

In the same way solve the following equations:

WRITTEN EXAMPLES.

1. Find the value of x when $\left\{\begin{matrix} 2x - 3y = 1 \\ y = 3 \end{matrix}\right\}.$

2. Find the value of y when $\left\{\begin{matrix} 6x + 2y = 10 \\ x = 1 \end{matrix}\right\}.$

3. Find the value of x when $\left\{\begin{matrix} 9x - 6y = 36 \\ y = 3 \end{matrix}\right\}.$

4. Find the value of x when $\left\{\begin{matrix} 5x - 7y = 6 \\ y = 2 \end{matrix}\right\}.$

56. *If two quantities be equal to each other, and we multiply them both by some other quantity, the two products must be equal. If we divide the two quantities by some other quantity, the two*

quotients will be equal. In the same way, if some quantity be added to or subtracted from each of the equal quantities, the two results in each case will be equal.

If $5x = 15$, we may multiply both members or parts by 2, and have $10x = 30$.

If $3x + 4y = 18$, we may multiply both members by 3, and have $9x + 12y = 54$.

If $-x = -6$, we may multiply both members by -1, and obtain $x = 6$.

If $-2x + 6y = -4$, we may multiply by -1, and obtain $2x - 6y = 4$.

Multiplying by -1 simply changes all the signs. As we may always multiply the members of an equation by -1, we may change all the signs of any equation, and still have the two members equal.

57. Find the value of x and y when

$$2x + 3y = 17,$$

and

$$4x - 3y = 7.$$

Solution. Here we cannot find the value of x or y, as before, by substituting the value of one of the letters, for we do not know it. If, however, we add together the corresponding members of the two equations, the terms containing y, having the same coefficients and different signs, cancel out.

$$\begin{array}{l} 2x + 3y = 17 \\ \underline{4x - 3y = 7} \end{array}$$

Thus adding, we have $6x = 24$

Dividing by 6, we find $x = 4$

Substituting the value of x in the first equation,

we have $8 + 3y = 17.$

Transposing and collecting, $3y = 9.$

Dividing by 3, we find $y = 3.$

Proof. Substituting these values in the first equation, we have

$$\begin{aligned} 8 + 9 &= 17; \\ 16 - 9 &= 7. \end{aligned}$$

Ex. Find the values of x and y when

$$\begin{aligned} 6x + 2y &= 46; \\ 6x - 4y &= 16. \end{aligned}$$

SOLUTION. Here, also, we have no value to substitute. If, however, we subtract the members of the second equation from the corresponding members of the first, it is evident that the terms containing x will cancel out.

$$\begin{array}{r} 6x + 2y = 46 \\ 6x - 4y = 16 \\ \hline \end{array}$$

Thus, subtracting, we have $6y = 30$

Dividing by 6, we find $y = 5$

Substituting the value of y in the first equation,

$$6x + 10 = 46.$$

Transposing and collecting, $6x = 36.$

Dividing by 6, we find $x = 6.$

Proof. Substituting the values of x and y in the first equations, we have

$$\begin{aligned} 36 + 10 &= 46; \\ 36 - 20 &= 16. \end{aligned}$$

58. If $x + 3y = 14,$
and $x = 5,$ on substituting the value of x the first equation becomes

$$5 + 3y = 14.$$

In this x does not occur, and the value of y can be readily found.

If $3x + 4y = 20,$

and $3x - 2y = 6,$ we have, on subtracting,

$$6y = 12.$$

In this equation x does not occur, and the value of y can be found.

The operation of combining two equations so as to get rid of one of the letters is called **Elimination.**

There are other methods of elimination, but the two above explained are those oftenest used. The first is called Elimination by **Substitution**; the second, Elimination by **Addition** or **Subtraction.**

59. In the following examples one of the letters has the same coefficient in each equation.

If the signs of these terms are unlike, add the corresponding members together. If the signs are alike, subtract them.

Note.—The pupil will bear in mind that in *subtracting* we change the signs in the subtrahend and add it to the minuend.

Find the value of x and y in the following equations:

WRITTEN EXAMPLES.

1. $\left\{\begin{array}{l} 2x + 5y = 24 \\ 2x + 2y = 12 \end{array}\right\}.$

2. $\left\{\begin{array}{l} 7x - 3y = 32 \\ 4x - 3y = 17 \end{array}\right\}.$

3. $\left\{\begin{array}{r} 8x + 3y = 65 \\ -8x - 9y = -83 \end{array}\right\}.$

4. $\left\{\begin{array}{l} 7x + 7y = 42 \\ 3x - 7y = -12 \end{array}\right\}.$

5. $\left\{\begin{array}{r} 10x + y = 32 \\ 3x + y = 18 \end{array}\right\}.$

6. $\left\{\begin{array}{r} x + 5y = 16 \\ 4x - 5y = 14 \end{array}\right\}.$

60. Given $2x + 5y = 22,$

$x - 2y = 2,$ to find the values of x and $y.$

Solution. Here the coefficients of neither of the letters are alike. If, however, we multiply the members of the second equation by 2, the coefficient of x in it will become equal to the coefficient of x in the other. The second will thus become

$$2x - 4y = 4;$$

and this, with the first equation repeated, will give us

$$2x + 5y = 22,$$

$$2x - 4y = 4.$$

These can now be solved as before.

Ex. Given $3x + 6y = 60$,

$4x + 5y = 56$, to find the values of x and y.

SOLUTION.—Here, as in the last example, we wish first to make the coefficients of one letter alike. We can make the coefficients of x alike by multiplying the members of the first equation by 4, and those of the second by 3. This will make both coefficients of x, 12. The equations will become

$$12x + 24y = 240,$$

$$12x + 15y = 168.$$

Subtracting, because the signs of the terms having the same coefficient are alike, we have

$$9y = 72.$$

Dividing by 9, we find $y = 8$.

Substituting this value in the first equation,

$$3x + 48 = 60.$$

Transposing and collecting, $3x = 12$.

Dividing by 3, we find $x = 4$.

Proof. Substituting these values in the first two equations,

$$12 + 48 = 60,$$

$$16 + 40 = 56.$$

61. The two coefficients of the same letter can always be made alike by multiplying the members of each equation by the coefficient of that letter in the other. Sometimes a smaller number is sufficient. If one term of an equation be multiplied, all the rest must also be multiplied by the same quantity.

We have therefore the following rule for solving equations containing two unknowns:

***Rule.**—I. Make the coefficients of one letter in both equations equal to each other by multiplying the members of one or both equations.*

II. If the signs of these terms are different, add the corresponding members. If the signs are alike, subtract them. This will eliminate one letter, and the value of the remaining one can be found by division.

III. Substitute this value in one of the equations to find the value of the other letter.

Find the values of x and y in each of the following pairs of equations:

WRITTEN EXAMPLES.

As a *Mental Exercise,* let the pupil tell by what numbers the members of the different equations are to be multiplied.

1. $\left\{\begin{array}{l} 4x - 3y = 10 \\ 2x - 3y = 2 \end{array}\right\}$,

2. $\left\{\begin{array}{l} 6x - 10y = 24 \\ 2x - 4y = 6 \end{array}\right\}$.

3. $\left\{\begin{array}{l} 7x + 5y = 42 \\ 3x - 8y = -53 \end{array}\right\}$.

4. $\left\{\begin{array}{l} 9x - 8y = 29 \\ 2x + 4y = 18 \end{array}\right\}$.

5. $\left\{\begin{array}{l} 12x - 10y = 56 \\ 7x - 5y = 31 \end{array}\right\}$.

6. $\left\{\begin{array}{l} 13x - 17y = -77 \\ 26x + 7y = -31 \end{array}\right\}$.

7. $\left\{\begin{array}{l} -4x - y = -36 \\ 7x + 2y = 64 \end{array}\right\}$.

8. $\left\{\begin{array}{l} x + y = 22 \\ x - y = 2 \end{array}\right\}$.

9. $\left\{\begin{array}{l} 3x - 2y = -5 \\ 2x - 3y = -10 \end{array}\right\}$.

10. $\left\{\begin{array}{l} 4x - 2y = 28 \\ 3x + 5y = 47 \end{array}\right\}$.

62. If the equations be not in the regular form for elimination, their terms must be transposed and arranged similarly in both equations.

Ex. Given $4x = 18 + 5y$,

and $4y - 29 = -3x$, to find x and y.

SOLUTION. Transposing, the equations become

$$4x - 5y = 18,$$
$$3x + 4y = 29.$$

These can now be solved as before.

Transpose, arrange similarly, and solve the following pairs of equations.

WRITTEN EXAMPLES.

1. $\begin{Bmatrix} 2x = 23 - y \\ 4x - 6 = 6y \end{Bmatrix}$.

2. $\begin{Bmatrix} 51 - 5y = 4x \\ 2x - 1 = y \end{Bmatrix}$.

3. $\begin{Bmatrix} 14 - 3y = -4x \\ x - 43 = -7y \end{Bmatrix}$.

4. $\begin{Bmatrix} 2x = 3y - 23 \\ 3x + 32 = 4y \end{Bmatrix}$.

5. $\begin{Bmatrix} 5x = 22 - 2y \\ 2x + 4 = 12y \end{Bmatrix}$.

6. $\begin{Bmatrix} 25 + 5y = 3x \\ 25y = 5 + 2x \end{Bmatrix}$.

SECTION XXI.

EXERCISES IN EQUATIONS OF TWO UNKNOWN QUANTITIES.

WRITTEN EXAMPLES IN THE FORMATION AND SOLUTION OF EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

1. IF 3 times one number, plus 4 times another, equals 10, and 5 times the first, minus 2 times the second, equals 8, what are the numbers?

NOTE.—In these examples form the equation without forming any intermediate expression.

STATEMENT. Let $x =$ one number,
and $y =$ the other.

Then, *because* 3 *times the first, plus* 4 *times the second, equals* 10, *we have the first equation,* $3x + 4y = 10$; *and because* 5 *times the first, minus* 2 *times the second, equals* 8, *we have the second equation.* (Here write the equation.)

2. If the sum of two numbers is 18, and the difference of them is 4, what are the numbers?

3. If the sum of two numbers is 20, and twice the first minus the second, is 1, what are the numbers?

4. If 4 times the cost of a pound of sugar, plus 5 times the cost of a pound of rice, is 76 cents, and 2 times the cost of a pound of sugar, plus 3 times the cost of a pound of rice, is 42 cents, how many cents a pound does each cost?

5. A grocer sends the following bill:

3 pounds of coffee and 2 pounds of tea,	$2.14
4 pounds of coffee and 3 pounds of tea,	3.07

How many cents per pound did he charge for each?

6. On a merchant's bill were the following items:

10 yards of calico and 6 yards of muslin,	$1.40
7 yards of calico, less 3 yards of muslin returned,	.26

What was the price per yard of each?

7. A butcher bought from one man 4 cows and 5 calves for $150. He bought from another 3 cows, and, after deducting the price of 2 calves which he sold him, he paid a balance of $78. For how many dollars each were the cows and calves sold?

MENTAL EXERCISE IN THE FORMATION OF EXPRESSIONS AND EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

1. If $x =$ number of pounds of tea purchased at 40 cents a pound, what represents the cost of the tea?

2. If a man buys x pounds of sugar at 10 cents a pound, and y pounds of coffee at 25 cents a pound, what expression represents the number of cents he has spent?

3. If a miller mixes x bushels of wheat of one kind with y bushels of another, and the mixture measures 16 bushels, what equation can be made?

4. In the case above, if the first wheat is worth $1 per bushel, and the second 90 cents per bushel, and the whole mixture is worth $15, what equation can be made?

5. If A has x dollars, and gives away $10, what expression represents the number he has left?

6. If B has y dollars, and receives $10, what represents the number he then has?

7. If A has x dollars, and B has y dollars, and B gives A $6, what expressions represent the amounts each have?

SECTION XXII.

SOLUTION OF PROBLEMS INVOLVING EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

1. A MAN worked 10 days and his son worked 6 days, when their united wages amounted to $13.80. At another time the man worked 6 days and the son 4, when the wages received were $8.40. How much did each receive for a day's work?

2. A man buys 12 pounds of sugar of two kinds for $1.04. One kind of sugar costs 6 cents, and the other 10. How many pounds of each did he buy?

3. A farmer takes 10 pounds of butter to a store, and receives for it 22 pounds of sugar and 2 cents. On another occasion he takes 6 pounds of butter, and receives 13 pounds

of sugar and 3 cents, the prices being the same as before. At what prices were the butter and sugar estimated?

4. A woman exchanged 9 dozen eggs for 7 yards of dress-goods, and received 3 cents change. But, wanting 5 yards more, she gave for them 6 dozen eggs and 3 cents. At what prices were the eggs and dress-goods reckoned?

5. A person has some balls of lead of one size, and some balls of iron of another. 3 balls of lead just balance 2 balls of iron and a 2-oz. weight, and 7 balls of lead and a 2-oz. weight will just balance 6 balls of iron. How many ounces does a ball of each weigh?

6. A and B had together $27. But when B gave to A $3, A had twice as much as B had left. How much had each at first?

7. A man valued his poultry, consisting of 30 chickens and 18 ducks, at $16.20. But a fox having killed 8 chickens and 6 ducks, what he had left was only worth $11.40. At what price per fowl did he rate them?

8. A man asked a miller to grind 12 bushels of corn and oats together, taking such an amount of each as would make the meal worth 40 cents a bushel, corn being worth 50 cents and oats 35. How many bushels of each should the miller put in?

9. A merchant has two kinds of oil—one worth 14 cents a gallon, and the other worth 24 cents. He wishes to fill a 40-gallon cask, so the mixture will be worth 18 cents a gallon. How much of each must he take?

10. A lady bought a piece of muslin at 12 cents a yard, and 6 yards of linen, the two pieces costing $2.40. At another time she bought twice as much of the same muslin, and 4 yards of linen costing twice as much as the former linen, the bill this time amounting to $4.00. How many yards of muslin were bought each time, and what were the prices of the linen?

11. A man engaged to be paid a certain sum per day for every day he worked, and to pay a certain sum for every day that he did not work. During the first 24 days he worked 20, and received $55. During the next 24 days he worked 14, and received $29.50. What were his wages, and what his board?

12. A person has two vessels whose capacity he wants to find. 5 measures of the smaller and 1 measure of the larger just fill a bushel; and if from 6 measures of the larger 2 measures of the smaller be taken, the result will fill a bushel. How many quarts will each of the vessels contain?

13. A seine on being drawn from the water was found to contain 2580 shad and herring, which were valued at $24, the shad being worth 10 cents apiece, and the herring 25 cents a hundred. How many of each kind were there, the herring being counted by even hundreds?

SUGGESTION.—Let x = the number of shad, and y = the number of hundreds of herring.

14. A works at a trench for 8 days, and B then finishes it in 5 days, and $22 is paid for the whole work. If A can do as much work, and therefore earn as much, in 6 days as B in 10 days, what part of the $22 should each receive?

SUGGESTION.—Let x = number of dollars per day A earned, and y = number B earned.

15. A carpenter earned $2.50 every day that he worked at his trade, and cut wood for $1.25 a day when he had no carpenter-work. At the end of 48 days he had earned $110. How many days was he employed at each kind of work?

16. At the end of one week a person paid wages to 7 men and 10 boys, amounting to $86. At the end of another week he paid, at the same rate, to 9 men and 12 boys, $108. How much per week did each earn?

SECTION XXIII.

DIVISION BY MONOMIALS.

DIVISION BY NUMBERS.

63. Ex. Divide $9ax$ by 3.

SOLUTION. Since $9ax$ means 9 times ax, one third of 9 times ax is 3 times ax, or $3ax$.

Division in Algebra is often indicated by writing the quantities in the form of a fraction, with the divisor for the denominator. Thus, $9ax$ divided by 3 would be expressed $\frac{9ax}{3}$, and would be read, $9ax$ *divided by* 3. In the same way, $\frac{12c^2}{4}$ indicates $12c^2$ *divided by* 4, and is so read.

Perform the division indicated in the following examples:

MENTAL EXERCISE.

1. $\frac{15a^2}{5}$.
2. $\frac{18abc}{6}$.
3. $\frac{36ax^2}{12}$.
4. $\frac{-27x^3}{9}$.
5. $\frac{6a^2y}{2}$.
6. $\frac{-40b^2}{10}$.

64. The quantity by which we divide is called the **Divisor**; the quantity divided is called the **Dividend**. The number of times which the divisor is contained in the dividend is called the **Quotient**.

Ex. Divide $27x^2$ by -3.

SOLUTION. Since the quotient shows how many times the divisor is contained in the dividend, the divisor multiplied by the quotient must give the dividend. Whenever a case of division is too difficult to perform at the first glance, we proceed to find what number we must multiply the divisor by to get the dividend. What then must we multiply -3 by to get $27x^2$? Remembering that in multiplying by a minus quantity we must change the sign of the product, we see that -3 must be multiplied by $-9x^2$; hence, $-9x^2$ is the quotient; that is, $\frac{27x^2}{-3} = -9x^2$.

Hence the following

***Rule.**—In dividing by a quantity with a minus sign, the sign of the quotient is minus when that of the dividend is plus, and is plus when that of the dividend is minus. That is, dividing by a minus quantity changes the sign of the term.*

ILLUSTRATION. $\frac{8x^2}{-4} = -2x^2$; $\frac{-25ax}{-5} = 5ax$.

Perform the division indicated in the following

MENTAL EXERCISE.

1. $\frac{4ax^3}{-2}$.
2. $\frac{-20y^2z}{-2}$.
3. $\frac{-8mn^2}{4}$.
4. $\frac{32abc}{-8}$.
5. $\frac{-75c^3d}{-25}$.
6. $\frac{12ay^3}{-6}$.

65. NOTE.—Where a polynomial is to be divided, divide each term separately, changing all the signs if the divisor has a $-$ sign.

ILLUSTRATION.

$$\frac{4ay + 2y}{2} = 2ay + y; \qquad \frac{14x^2 - 21y^2}{-7} = -2x^2 + 3y^2.$$

7. $\frac{25a - 20b + 15c}{-5}$.
8. $\frac{-90ax - 54x^2 + 27xy}{9}$.
9. $\frac{24x^2 - 48y^2 - 96z^2}{-12}$.
10. $\frac{42c^2 - 36cd + 18d^2}{6}$.
11. $\frac{49b^2 - 56bc + 21c^2}{-7}$.
12. $\frac{48abc - 60ab^2 + 12c^3}{6}$.

DIVISION BY LETTERS.

66. Divide *abc* by *ab*.

Solution. Remembering that the quotient is that quantity by which the divisor must be multiplied in order to give the dividend, we see at once that *ab* must be multiplied by *c* to give *abc*. Hence, *c* is the quotient. So in every case we must multiply the divisor by those letters of the dividend which the divisor lacks; that is, by those letters which would be left in the dividend after striking out those of the divisor.

Ex. Divide *cdxy* by *cx*.

Solution. Striking out *c* and *x* from the dividend, we have *dy* for the quotient.

Ex. Divide $15\,mny$ by $3\,y$.

Solution. Divide first by the 3; we have $5\,mny$; then by *y*, and we have $5\,mn$.

The result would have been the same had we divided 15 by 3 and *mny* by *y*. Hence the

Rule.*—In dividing by a monomial, divide the coefficients first, and write after the result the letters which are left of the dividend after striking out those of the divisor. If the divisor have a minus sign, the sign of the quotient will be changed from that of the dividend.*

Perform the division indicated in the following

MENTAL EXERCISE.

1. $\dfrac{7\,a^3x - 4\,bx + 5\,x}{x}$.
2. $\dfrac{14\,bcd - 16\,acd + 18\,cdx}{2\,cd}$.
3. $\dfrac{-16\,abcy + 20\,axy - 40\,axyz}{-4\,ay}$.
4. $\dfrac{12\,acm - 9\,cdmx + 3\,cm}{3\,cm}$.
5. $\dfrac{45\,abcn - 15\,bdnx + 10\,bny}{5\,bn}$.
6. $\dfrac{-24\,abc - 42\,acy + 12\,ac}{-6\,ac}$.

67. Divide a^7 by a^3.

SOLUTION. The exponent 7 in the dividend indicates 7 a's multiplied together. The exponent 3 in the divisor indicates 3 a's multiplied together; hence, striking out 3 of the 7 a's in the dividend, we will have 4 a's for the quotient, or a^4.

The *number* of letters in the divisor is subtracted from the *number* of letters in the dividend to find the *number* in the quotient. But the exponents show the numbers. Hence the

***Rule.**—When the same letter occurs in both dividend and divisor, subtract its exponent in the divisor from its exponent in the dividend to find its exponent in the quotient.*

ILLUSTRATION.

$$\frac{a^4}{a^2} = a^2, \quad \frac{4ax^3}{2ax} = 2x^2, \quad \text{and} \quad \frac{20x^2y - 12xy^2}{4xy} = 5x - 3y.$$

NOTE.—Remember that when no exponent is written, 1 is understood. Thus, $x = x^1$.

Perform the division indicated in the following

MENTAL EXERCISE.

1. $\dfrac{a^9}{a^5}$.

2. $\dfrac{7ax^5}{x^3}$.

3. $\dfrac{18b^2c^3x}{-3bc^2x}$.

4. $\dfrac{25x^3y^2 - 30ax^2y}{5x^2y}$.

5. $\dfrac{-14abc^2 + 42ac^4d}{-7ac^2}$.

6. $\dfrac{16m^2n - 32m^4}{8m^2}$.

7. $\dfrac{4c^4dx - 7a^2c^3d}{c^2d}$.

8. $\dfrac{-9x^5 + 10x^4 - 6x^3}{-x^3}$.

9. $\dfrac{24a^4bx^2 - 60a^3x^4y}{-12a^3x^2}$.

10. $\dfrac{-48b^6 - 12b^5c + 6b^4c^2}{6b^4}$.

11. $\dfrac{-a^3c^3x^3 + a^4c^2x^5 - a^5c^3x^6}{-a^3c^2x}$.

12. $\dfrac{am^4n^4y - bm^6n^2y^3}{m^4n^2y}$.

13. $\dfrac{9\,b^9n^6 - 12\,a^3b^6n^8 + 6\,b^5c^2n^5}{3\,b^5n^4}$.

14. $\dfrac{28\,x^5yz^2 - 16\,x^4y^3z^2 + 4\,x^4yz^4}{4\,x^4yz^2}$.

15. $\dfrac{-\,14\,c^4d^2y^3 + 35\,cdy^4 - 21\,cy^2}{-\,7\,cy^2}$.

16. $\dfrac{18\,b^3n^3y^3 - 54\,b^6n^2y^3 + 36\,b^3n^4y^4}{9\,b^3ny^2}$.

17. $\dfrac{-\,18\,a^2c^2x^2y^2 + 24\,ac^3x^5y^7 - 10\,a^3c^2x^3y^3}{-\,2\,ac^2xy^2}$.

18. $\dfrac{-\,2\,a^3c^3dy^4 + c^2dy^3 - 8\,c^2dxy^3}{-\,c^2dy^3}$.

SECTION XXIV.

ARRANGEMENT OF TERMS.

68. It is often convenient to have the terms of a polynomial *regularly* arranged. This is done when the terms are placed in the order of the exponents of one of the letters. Thus, $x^3 + x^2 + x + 1$ is arranged with the term containing the highest exponent of x first, and the others regularly following. $1 + y + y^2 + y^3$ is regularly arranged, because the term containing the lowest exponent comes first and the others follow in order. The term containing the highest exponent is generally placed first.

Give the regular order for the terms of the following polynomials.

MENTAL EXERCISE.

1. $x + 2\,x^3 - 3\,x^2$.
2. $y^2 + y^4 - 3\,y - 5\,y^3$.
3. $a^3 - 4\,a + 7\,a^2$.
4. $b^6 + b - 3\,b^4 + 5\,b^2$.
5. $x^2 - 4\,x^6 + 6\,x^4 - 2\,x^8 + x^{10}$.
6. $6\,y^3 - 7\,y + 4\,y^5$.

69. When two or more letters occur in each term, the terms are arranged with reference to the exponents of one of the letters.

7. Arrange with reference to a, $2ax + a^2 + x^2$.

8. Arrange with reference to x, $3x^2y + x^3y^3 + 3xy^2$.

9. Arrange with reference to c, $c^2x + cx^2 + c^3 + x^3$.

10. Arrange with reference to a, the two polynomials,
$$x + a \text{ and } 3a^2x + a^3 + x^3 + 3ax^2.$$

Arrange with reference to the same letter, the terms of the two polynomials in each of the following examples:

11. $a + b$ and $b^2 + 2ab + a^2$.

12. $2xy + y^2 + x^2$ and $x^3 + y^3 + 3x^2y + 3xy^2$.

13. $a^2 + x^2 - ax$ and $x^3 + a^3$.

14. $a^2 - b$ and $b^2 + a^4 - 2a^2b$.

15. $-2 + 2x$ and $-2x^3 - 6x + 2 + 6x^2$.

16. $a^2 + x^2 + ax$ and $a^4 + x^4 + a^2x^2$.

SECTION XXV.

REVIEW OF SUBTRACTION.

Minuend,	$10ax$
Subtrahend,	$6ax$
Remainder,	$4ax$

70. If the pupil finds it difficult to remember which part is the minuend and which is the subtrahend, let him think that Minuend means, *that which is diminished*, and Subtrahend, *that which is subtracted.* The similarity between *subtracted* and *subtrahend* should make that part easily remembered.

The *Remainder* is that quantity which must be added to the subtrahend, or subtracted quantity, to give the minuend, or diminished quantity. As shown in Art. 28, it may always be found by adding the subtrahend, with its sign changed, to the minuend.

We add as if the signs were changed, and do not actually change them.

In performing the addition, only similar terms can be collected into one. When the terms are not similar, we must connect them with their proper signs.

Subtract the following:

MENTAL EXERCISE.

1. $$\begin{array}{r} x + y \\ \underline{x - y} \end{array}$$

2. $$\begin{array}{r} x - y \\ \underline{x + y} \end{array}$$

3. $$\begin{array}{r} a^2x \\ \underline{\quad\quad ax^3} \end{array}$$

4. $$\begin{array}{r} b \\ \underline{\quad a \quad} \end{array}$$

5. $$\begin{array}{r} x^2 + \quad ax + x^3 \\ \underline{x - 2ax + x^3} \end{array}$$

6. $$\begin{array}{r} a^3 - b^3 \\ \underline{a^3 - a^2b} \end{array}$$

7. $$\begin{array}{l} a^3 - b^3 \\ \underline{a^3 + a^2b + ab^2} \end{array}$$

8. $$\begin{array}{l} a^2 + 2ab + b^2 \\ \underline{a^2 + \quad ab \quad\quad\;} \end{array}$$

9. $$\begin{array}{r} 7a^2 - 8ay + 9y^2 \\ \underline{3a^2 - 7ay - 2y^2} \end{array}$$

WRITTEN EXAMPLES.

1. Subtract $2x + 3$ from $6x + 9$, and from the remainder subtract again $2x + 3$, continuing that operation until there is no remainder.

2. Subtract $4a^2 - 2b^2$ from $16a^2 - 8b^2$, and repeat the subtraction until there is no remainder.

3. Subtract $2cd + 3x$ from $4cd + 6x$, and continue the subtraction until there is no remainder.

4. Subtract $a^2 - 2x^2$ from $3a^2 + 2x^2$, and continue the subtraction until there is no term containing a^2.

5. Multiply $2x + 3$ by 3, and subtract the product from $6x + 9$.

6. Multiply $4a^2 - 2b^2$ by 4, and subtract the product from $16a^2 - 8b^2$.

7. Multiply $2cd + 3x$ by 2, and subtract the product from $4cd + 10x$.

8. Multiply $a^2 - 2x^2$ by 3, and subtract the product from $3a^2 + 2x^2$.

9. Multiply $2c + 3d$ by such a number that the product, on being subtracted from $10c - 2d$, will cancel out the first term.

10. Multiply $6b^2 - 4$ by such a number that the product, on being subtracted from $18b^2 + 4$, will cancel out the first term.

11. Multiply $4xy - 3y^2$ by such a number that the product, subtracted from $24xy - 18y^2$, will cancel out the first term.

12. Multiply $a + x$ by a, and subtract the product from $a^2 + 2ax + x^2$.

13. Multiply $a + x$ by x, and subtract the product from $ax + x^2$.

14. Multiply $x + y$ by such a letter that the product, on being subtracted from $x^2 + 2xy + y^2$, will cancel out the first term.

15. Multiply $x + y$ by such a letter that the product, on being subtracted from $xy + y^2$, will cancel out the first term.

16. Multiply $a + b$ by such a letter that the product, on being subtracted from $a^2 - b^2$, will cancel out the first term.

17. Multiply $a + b$ by $-b$, and subtract the product from $-ab - b^2$.

18. Multiply $a^2 + ax + x^2$ by a, and subtract the product from $a^3 - x^3$.

SECTION XXVI.

DIVISION (Continued).

71. One quantity contains a less quantity as many times as the smaller may be taken away, or subtracted, from it.

In Division the quotient shows how many times the dividend contains the divisor. Hence, we may obtain the quotient by finding how many times the divisor may be subtracted from the dividend. This is the method in all *Long Division.*

In dividing one polynomial by another, as $a^2 + 2ax + x^2$ by $a + x$, we take at each subtraction not the divisor itself, but that number of times the divisor which will cancel out the first term of the dividend; that is, we multiply the divisor by such a number or quantity as will make its first term equal to the first term of the dividend, and then subtract this product. By cancelling out the terms one after another in this way, by successive subtractions we will finally obtain the smallest possible remainder. By adding together the quantities by which the divisor was multiplied, or taken, at each subtraction, we will find the whole number of times that the divisor was taken from the dividend. This will be the quotient.

The multiplier in each case may be found by dividing the first term of the divisor into the first term of the dividend.

72. Divide $6x + 2$ by $3x + 1$.

Solution. Writing the example in the usual form for long division, we divide $3x$ into $6x$ to find what number of times $3x + 1$ must be subtracted from $6x + 2$ to make the first terms cancel. $3x$ is contained in $6x$ 2 times. Placing the 2 in the place for the quotient, multiplying the divisor by it, and subtracting the product from the dividend, we find no remainder. The whole operation would be—

$$\begin{array}{rll}
\textbf{Divisor.} & \textbf{Dividend.} & \textbf{Quotient.}\\
3x+1\,) & 6x+2\,(& 2\\
 & \underline{6x+2} &
\end{array}$$

Hence the quotient is 2, because 2 times the divisor subtracted from the dividend leaves no remainder.

Ex. Divide $12b^2+16a$ by $4b^2+2a$.

Solution. Writing the quantities as before, finding the proper multiplier by dividing the first term of the divisor into the first term of the dividend, multiplying the divisor by it, and subtracting, the work stands thus:

$$\begin{array}{rlll}
\textbf{Divisor.} & \textbf{Dividend.} & \textbf{Quotient.} & \\
4b^2+2a\,) & 12b^2+16a\,(& 3 & \\
 & \underline{12b^2+6a} & & \\
 & 10a & \textbf{Remainder.} &
\end{array}$$

We have here a remainder of $10a$. This is as simple a remainder as we can get; so we indicate the division of it by writing it over the divisor, as a fraction, the whole quotient being $3+\dfrac{10a}{4b^2+2a}$.

Ex. Divide $ax+b$ by $x+y$.

Solution. Performing the operation as before, we have

$$\begin{array}{rll}
\textbf{Divisor.} & \textbf{Dividend.} & \textbf{Quotient.}\\
x+y\,) & ax+b\;\;(& a+\dfrac{b-ay}{x+y}\\
 & \underline{ax+ay} & \\
 & b-ay &
\end{array}$$

In the same way perform the following

WRITTEN EXAMPLES.

1. Divide $10c^2+5d$ by $2c^2+d$.
2. Divide $18ab+9b^2$ by $6ab+2b^2$.
3. Divide $ax+x^2$ by $a+x$.

4. Divide $xy + y^2$ by $x + y$.
5. Divide $a^2 + ax$ by $a + x$.
6. Divide $3a^2 + 3ax$ by $a + x$.
7. Divide $x^3 + x^2y$ by $x + y$.
8. Divide $4ab^2 + 9b$ by $2ab + 3$.
9. Divide $2xy + y^2$ by $2x - y$.
10. Divide $6cd + d^2$ by $3c - 4d$.
11. Divide $4x + 4y$ by $x + y$.
12. Divide $6x^2 + 6x$ by $2x^2 + 2x$.

73. Divide $x^2 + y^2 - 2xy$ by $x - y$.

OPERATION.

Divisor. Dividend. Quotient.

$$\begin{array}{rl}
x - y\,) & x^2 - 2xy + y^2\,(\,x - y \\
 & \underline{x^2 - \;\;xy} \\
 & \quad\;\; - \;\;xy + y^2 \\
 & \quad\;\; \underline{- \;\;xy + y^2}
\end{array}$$

Explanation. We must first arrange the terms regularly with reference to the exponents of x. This will put the dividend in the form $x^2 - 2xy + y^2$. Then, writing the example, and proceeding as before, we have the first remainder, $-xy + y^2$. Now, since x, the first term of the divisor, is contained $-y$ times in $-xy$, the first term of the remainder, we multiply the divisor by $-y$, and subtract the product from the remainder. This cancels out its first term. Performing the operation, we find that the last terms also cancel out; hence there is no remainder. We have taken x times the divisor, $x - y$, and $-y$ times $x - y$ from the dividend, and have no remainder; hence the dividend contains the divisor exactly x and $-y$ times, or $x - y$ times.

Ex. Divide $x^3 + y^3$ by $y + x$.

Explanation. Arranging the terms of the divisor and dividend regularly with reference to the exponents of the same letter, we have the divisor in the form of $x + y$. Proceeding as before, we have the following result:

$$
\begin{array}{llll}
\text{Divisor.} & \text{Dividend.} & & \text{Quotient.} \\
x + y) & x^3 + y^3 & (& x^2 - xy + y^2 \\
 & \underline{x^3 + x^2y} & & \\
 & \quad - x^2y + y^3 & & \\
 & \quad \underline{- x^2y - xy^2} & & \\
 & \qquad\quad + xy^2 + y^3 & & \\
 & \qquad\quad \underline{+ xy^2 + y^3} & &
\end{array}
$$

The quotient is $x^2 - xy + y^2$, because the divisor has been subtracted x^2 and $- xy$ and y^2 times from the dividend, leaving no remainder.

The preceding principles are contained in the following

Rule.—*I. Arrange the terms of both divisor and dividend according to the exponents of the same letter.*

II. Divide the first term of the divisor into the first term of the dividend. This will give one term of the quotient.

III. Multiply the divisor by this term, and subtract the product from the dividend.

IV. The remainder thus obtained will make a new dividend, and the operation will be continued till the first term of the dividend will not contain the first term of the divisor.

V. Write the last remainder over the divisor in the form of a fraction, and add it to the quotient.

WRITTEN EXAMPLES.

1. Divide $a^2 + 2ab + b^2$ by $a + b$.
2. Divide $c^2 - 2cd + d^2$ by $c - d$.
3. Divide $a^2 - b^2$ by $a - b$.
4. Divide $x^2 - y^2$ by $x + y$.
5. Divide $x^3 - 3x^2y + 3xy^2 - y^3$ by $x - y$.
6. Divide $4a^2 + 4a + 1$ by $2a + 1$.

7. Divide $x^4 - y^4$ by $x - y$.

8. Divide $x^3 - y^3$ by $x^2 + xy + y^2$.

9. Divide $a^4 - 2a^2b + b^2$ by $a^2 - b$.

10. Divide $6x^3 - 2x^2 - 6x + 2$ by $2x - 2$.

Note.—The pupil will look carefully to the arrangement of the following examples before attempting to divide:

11. Divide $13x + 15x^3 - 17x^2 - 3$ by $5x^2 - 4x + 3$.

12. Divide $8a^3 - 16a^2b - 4ab^2 + 4b^3$ by $2a - 2b$.

13. Divide $6cd^3 - 2c^3d - 6c^2d^2 - 2d^4 + 12c^4$ by $2cd + 4c^2 - 2d^2$.

14. Divide $2a^2 - ab - 2ac - 6b^2 + 4bc$ by $a - 2b$.

15. Divide $a^3 + x^3$ by $a - x$.

16. Divide $a^4 + x^4$ by $a + x$.

17. Divide $a^4 + a^2x^2 + x^4$ by $a^2 + ax + x^2$.

18. Divide $ax + bx + cx + ay + by + cy$ by $a + b + c$.

19. Divide $m^4 + m^5 - 2m - 2m^2$ by $m^2 + m$.

SECTION XXVII.

THEOREMS.

74. **The Square** of a quantity is the product obtained by multiplying the quantity by itself.

Ex. What is the square of $a + b$?

OPERATION.

$$\begin{array}{r} a + b \\ \underline{a + b} \\ a^2 + ab \phantom{{}+b^2} \\ \underline{ab + b^2} \\ a^2 + 2ab + b^2 \end{array}$$

Explanation. Performing the multiplication, we find for the product $a^2 + 2ab + b^2$.

Hence, the square of $a + b$, which is the sum of the two quantities a and b, equals the square of a (the first quantity), plus twice the product of a and b, or (the first and second quantities) plus the square of b (the second quantity).

This may be more briefly expressed in the following theorem:

NOTE. The following theorems must be carefully learned, word for word:

***Theorem I.**—The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first and second, plus the square of the second.*

75. What is the square of $a - b$?

OPERATION.

$$\begin{array}{lcr} a & - & b \\ a & - & b \\ \hline a^2 & - & ab \\ & - & ab + b^2 \\ \hline a^2 & - & 2ab + b^2 \end{array}$$

EXPLANATION. Performing the multiplication, we find for the product $a^2 - 2ab + b^2$.

Here the square of $a - b$, which is the difference of the two quantities a and b, is equal to the square of a (the first quantity), minus twice the product of a and b (the first and second quantities), plus the square of b (the second quantity).

This may be more briefly expressed thus:

***Theorem II.**—The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first and second, plus the square of the second.*

76. What is the product of $a + b$ and $a - b$?

OPERATION.

$$\begin{array}{lcr} a & + & b \\ a & - & b \\ \hline a^2 & + & ab \\ & - & ab - b^2 \\ \hline a^2 & & - b^2 \end{array}$$

EXPLANATION. Performing the multiplication, we have as the product $a^2 - b^2$.

Hence the product of $a + b$ (which is the sum of the two quantities a and b) and $a - b$ (which is the difference of the two quantities a and b) is equal to $a^2 - b^2$, or the difference of the squares of the two quantities.

This may be more briefly expressed thus:

***Theorem III.**—The product of the sum and difference of two quantities is equal to the difference of their squares.*

These Theorems are useful in abridging the work of multiplication and division.

SECTION XXVIII.

USE OF THEOREMS IN MULTIPLICATION.

77. Ex. What is the square of $x + y$?

SOLUTION. $x + y$ is the sum of two quantities, x and y. By the use of Theorem I. we can find its square directly without multiplying it out. It will be equal to the square of the first quantity, or x^2, plus twice the product of the first and second, or $2xy$, plus the square of the second, or y^2. That is, it equals $x^2 + 2xy + y^2$.

In the same way, by direct use of Theorem I., perform the following examples:

MENTAL AND WRITTEN EXERCISE.

1. What is the square of $m + n$?
2. What is the square of $2 + x$?
3. What is the square of $b + 3$?
4. What is the square of $2a + b$?
5. What does $(c + d)^2$ equal?
6. What does $(4 + y)^2$ equal?
7. What does $(a + x)(a + x)$ equal?

NOTE.—$a + x$ multiplied by $a + x$ equals $(a + x)^2$.

8. What does $(5x + 2y)(5x + 2y)$ equal?

Ex. What is the square of $c - d$?

SOLUTION. Since $c - d$ is the difference of two quantities, its square may be found directly by Theorem II. without the actual multiplication. It will be the square of the first quantity, or c^2, minus twice the product of the first and second, or $-2cd$, plus the square of the second, or d^2; that is, it equals $c^2 - 2cd + d^2$.

In the same way perform the following examples:

MENTAL AND WRITTEN EXERCISE.

1. What is the square of $a - b$?
2. What is the square of $x - y$?
3. What does $(m - n)^2$ equal?
4. What does $(3 - x)^2$ equal?
5. What does $(2 - y)(2 - y)$ equal?
6. What does $(3a - b)(3a - b)$ equal?

Ex. What does $(x + y)(x - y)$ equal?

SOLUTION. We want here the product of the sum and difference of the two quantities, x and y, $x + y$ being the sum, and $x - y$ being the difference. This product, according to Theorem III., is equal to the difference of their squares; that is, equal to $x^2 - y^2$.

MENTAL AND WRITTEN EXERCISE.

1. What is the product of $c + d$ and $c - d$?
2. What is the product of $m + n$ and $m - n$?
3. What does $(a^2 - b)(a^2 + b)$ equal?
4. What does $(3 - x)(3 + x)$ equal?
5. What does $(y + 4)(y - 4)$ equal?
6. What does $(x^2 - x)(x^2 + x)$ equal?
7. What does $(3a + 2b)(3a - 2b)$ equal?
8. What does $(5y^2 + 4z^2)(5y^2 - 4z^2)$ equal?

78. The two expressions $(a + b)^2$ and $(a + b)(a + b)$ mean exactly the same. Instead of writing $a + b$ twice, we may write it once, and affix the exponent 2 to show that it is to be taken twice. Both methods indicate the square of $a + b$. In expanding such expressions use Theorem I.

Whenever expressions like $(m - n)(m - n)$, or $(m - n)^2$

are to be expanded we use Theorem II., because they indicate the square of the *difference* of two quantities.

In expressions like $(x - 4)(x + 4)$, where the two terms are the same in each parenthesis, but are connected by + in one and by − in the other, use Theorem III., because the product of the sum and difference is indicated.

In performing the following examples, first observe which Theorem should be used, then derive the result from that directly, without actual multiplication.

Expand the following expressions:

MENTAL AND WRITTEN EXERCISE.

1. $(x + y)^2$.
2. $(3a + b)(3a + b)$.
3. $(2x - 2y)(2x - 2y)$.
4. $(6 - x)(6 + x)$.
5. $(4a - 2b)(4a - 2b)$.
6. $(a^2 + b^2)^2$.
7. $(x^2 - x)(x^2 - x)$.
8. $(5y^2 - 2z^2)(5y^2 + 2z^2)$.
9. $(7 - 2b)^2$.
10. $(4a + 3b)(4a + 3b)$.
11. $(2ac - cd)^2$.
12. $(4xy + y^2)^2$.
13. $(3xy + y^2)(3xy - y^2)$.
14. $(2 - x)(2 + x)$.
15. $(4 - x^2)(4 + x^2)$.
16. $(mn - cd)(mn - cd)$.
17. $(x^3 + x^2)^2$.
18. $(3abc^2 - 2ac^2)^2$.
19. $(x^3 - x)(x^2 + x)$.
20. $(2ax - 3x^2)(2ax - 3x^2)$.
21. $(3x - 4y)^2$.
22. $(a - ab)(a + ab)$.
23. $(ax - 2x^2)(ax - 2x^2)$.
24. $(2a + 5b)(2a + 5b)$.
25. $(m^2 - n^2)(m^2 + n^2)$.
26. $(x^3 - y^3)(x^3 + y^3)$.
27. $(a^3 + x^3)^2$.
28. $(axy + xyz)^2$.
29. $(4ab + 2b^2)^2$.
30. $(mn - n^2)(mn - n^2)$.
31. $(2x + 4y)(2x - 4y)$.
32. $(1 - 3x)(1 - 3x)$.
33. $(4y + 1)^2$.
34. $(6a - 1)(6a + 1)$.

SECTION XXIX.

USE OF THEOREMS IN DIVISION.

79. The foregoing Theorems also enable us, in many cases, to shorten the work of Division.

Ex. Divide $a^2 + 2ab + b^2$ by $a + b$.

Solution. We know by Theorem I. that the square of $a + b$ is $a^2 + 2ab + b^2$. Therefore, $a + b$ is contained in $a^2 + 2ab + b^2$ $a + b$ times. It might be expressed thus:

$$\frac{a^2 + 2ab + b^2}{a + b} = a + b.$$

Ex. What does $\frac{x^2 - y^2}{x + y}$ equal?

Solution. We know by Theorem III. that $x + y$ multiplied by $x - y$ gives $x^2 - y^2$. Hence $\frac{x^2 - y^2}{x + y} = x - y$.

Give the results of the following indicated divisions:

MENTAL EXERCISES.

Solve by Use of Theorem I.

1. $\frac{x^2 + 2xy + y^2}{x + y}$.
2. $\frac{4a^2 + 4ab + b^2}{2a + b}$.
3. $\frac{9c^2 + 24cd + 16d^2}{3c + 4d}$.
4. $\frac{25x^2 + 20xy + 4y^2}{5x + 2y}$.

Solve by Use of Theorem II.

5. $\frac{b^2 - 2bc + c^2}{b - c}$.
6. $\frac{9m^2 - 6mn + n^2}{3m - n}$.

SOLVE BY USE OF THEOREM III.

7. $\dfrac{a^2 - b^2}{a - b}$.

8. $\dfrac{4 - x^2}{2 + x}$.

9. $\dfrac{16y^2 - 25z^2}{4y - 5z}$.

10. $\dfrac{x^8 - x^4}{x^4 + x^2}$.

Solve by whichever Theorem is applicable to the example:

11. $\dfrac{a^2 - 6ab + 9b^2}{a - 3b}$.

12. $\dfrac{x^2 - 9}{x + 3}$.

13. $\dfrac{a^2b^2 - c^2d^2}{ab + cd}$.

14. $\dfrac{4m^2 - 4mn + n^2}{2m - n}$.

15. $\dfrac{25x^2 + 10xy + y^2}{5x + y}$.

16. $\dfrac{4x^2 - a^2b^2}{2x - ab}$.

17. $\dfrac{49x^2 + 70xy + 25y^2}{7x + 5y}$.

18. $\dfrac{m^4 - n^2}{m^2 + n}$.

19. $\dfrac{x^4 - 2x^3 + x^2}{x^2 - x}$.

20. $\dfrac{x^8 - x^4}{x^6 + x^2}$.

21. $\dfrac{x^8 - y^8}{x^4 + y^4}$.

22. $\dfrac{x^2 - 2xy + y^2}{x - y}$.

SECTION XXX.

FACTORING.—MONOMIAL FACTORS.

80. THE **Factors** of a quantity are such quantities as, when multiplied together, will give that quantity. Thus, the factors of 6 are 2 and 3; of xy are x and y; of a^2b are a, a, and b.

Any **divisor** of a quantity is a **factor** of it.

Factoring is the process of finding the factors of any quantity.

A Prime Quantity is one which has no factors.

NOTE.—A *monomial*, it will be remembered, is a quantity having no parts connected by signs, + or −.

A *polynomial* is an expression having two or more parts connected by signs, + or −.

MENTAL EXERCISES IN FACTORING.

Give the prime factors of the following numbers:

1. 4. 2. 6. 3. 21. 4. 8. 5. 12.

Give the letters which have been multiplied together to produce the following monomials:

6. abc. 7. a^3b. 8. c^3. 9. c^2x^2. 10. x^2y.

Give the prime factors of the following monomials:

11. $6a^2x$. 12. $8xy^2$. 13. $15m^2n^2$. 14. $12d^4$. 15. $18a^3x$.

What letter will divide both terms of the following expressions?

1. $a^2 + ax$. 3. $2xy + y^2$. 5. $m^2n + n$.
2. $x^2 + 2xy$. 4. $ab + ac$. 6. $x - x^2$.

What *monomial* will divide all the terms in each of the following expressions?

7. $3xy^2 + 3y^3$. 9. $6a^3 + 13a^2y$. 11. $x^3y - x^2y^3$.
8. $5m^3n - mn^4$. 10. $a^2x - ax^2$. 12. $3b^4 - 6a^2b^3$.

What is the *greatest monomial* that will divide all the terms of each of the following expressions? and what is the result of the division?

13. $ax + ay + az$. 15. $2xy + 2y^2$.
14. $a^4b^2 - a^2b^4$. 16. $4a^2 + 4$.

In each of the following expressions find first the monomial divisor or factor, and then by division, as before, find the other factor:

MENTAL AND WRITTEN EXERCISES.

1. $a^2 + ay$.
2. $2a + 2x$.
3. $3cd + 3d^2$.
4. $4x^2 - 8xy$.
5. $abc - ab^2$.
6. $a^2x - 3ax^2$.
7. $cd + cx + cy$.
8. $2a^2 + 2ab + 2ax$.
9. $a^3 + 2a^2b - ab^2$.
10. $3x^2y - 3xy^2 + y^3$.
11. $6ab - 12ac + 9ay$.
12. $a^2b^3c^2 + bc^2xy$.
13. $8x^3y^2z - 14xy^2z^2$.
14. $12c^3d^3 - 4c^2d^3$.
15. $50ax^2y + 70axy^2$.
16. $12m^3n^2 + 24m^2n^3 - 60m^2n^2x$.

SECTION XXXI.

FACTORING.—POLYNOMIAL FACTORS.

81. If we have to factor a large *number*, such as 615 or 832, we begin by dividing by some number that we see will go into it; that is, by 5 in the first case, and by 2 in the second. But if we have to factor a number like 9 or 15 or 35, we do not look for divisors, but think at once that 3 × 3 = 9, 3 × 5 = 15, and 5 × 7 = 35. The two factors are evident at the same time as soon as we see the number.

There are certain kinds of algebraic expressions which we must learn to factor in the same way.

82. We know by Theorem I. that $(a + b)(a + b)$ equals $a^2 + 2ab + b^2$. Hence, the factors of $a^2 + 2ab + b^2$ are $a + b$ and $a + b$.

Also, from Theorem II. we know that $x^2 - 2xy + y^2$ is the square of $x - y$, and therefore that its factors are $x - y$ and $x - y$.

An expression like $a^2 + 2ab + b^2$, or $x^2 - 2xy + y^2$, which is the square of another expression, is called a perfect square. In both these perfect squares,

I. There are three terms.

II. The first and last terms are each squares, with the sign +.

III. And if twice the square root of the first term be divided into the second term, and the quotient squared, the result will be the third term.

If the sign of the middle term be +, the expression is to be factored by Theorem I.; if −, by Theorem II.

Ex. Is $x^2 + 4x + 4$ a perfect square?

Solution. We see, first, that it has three terms; next, that its first and last terms are both squares, and have the sign +; and lastly, that twice the square root of its first term divided into the second gives 2, the square of which is 4, which is the third term.

The quantity therefore is a perfect square, and its factors are $x + 2$ and $x + 2$.

Ex. Is $4x^2 - 6xy + 9y^2$ a perfect square?

Solution. We see, first, that it has three terms; next, that its first and last are squares and have + signs; lastly, we see that twice the square root of its first term divided into the middle term gives $\frac{-3y}{2}$, the square of which is not equal to the third term. Hence, it is not a perfect square.

Tell which of the following are perfect squares, and give the factors of those which are:

MENTAL EXERCISE.

1. $c^2 - 2cd + d^2$.
2. $c^2 + 2c + 1$.
3. $4d^2 - 4dy + y^2$.
4. $m^2 + 2mn + n^2$.
5. $a^2 + ab + b^2$.
6. $a^2 + 2ab - b^2$.
7. $4x^2 + 12xy + 9y^2$.
8. $25 + 10x + x^2$.
9. $25 - 20x + 16x^2$.
10. $x^4 - 4x^2y + 4y^2$.
11. $a^4 - 2a^3 + a^2$.
12. $x^6 + 2x^5 + x^4$.

13. $4 + 4xy + x^2y^2$.
14. $16x^2 - 8xy + y^2$.
15. $4a^2 - 2ax + x^2$.
16. $9a^3 - 12a^2 + 4a$.

83. According to Theorem III., the product of $a + x$ and $a - x$ is $a^2 - x^2$. Hence, the factors of $a^2 - x^2$ are $a + x$ and $a - x$. Any *Binomial* which is the difference of two squares can be factored according to Theorem III.

MENTAL EXERCISE.

1. What quantities multiplied together will give $c^2 - d^2$?
2. What quantities multiplied together will give $a^2 - b^2$?
3. What quantities multiplied together will give $m^2 - 1$?
4. What quantities multiplied together will give $a^4 - b^2$?
5. What quantities multiplied together will give $x^6 - y^6$?

What are the factors of the following?

6. $a^2b^2 - c^2$.
7. $m^2 - n^2x^2$.
8. $1 - x^2$.
9. $9x^2 - 4y^2$.
10. $16a^2b^2 - x^2$.
11. $25 - a^4$.
12. $a^6 - b^8$.
13. $a^2 - 49b^2$.
14. $x^2y^4 - z^6$.
15. $4a^2b^6 - 25x^4$.

Tell which of the following trinomials are perfect squares, and factor those which are. Tell which of the binomials are the differences of two squares, and factor those which are.

MENTAL EXERCISE.

1. $b^2 + 2bx + x^2$.
2. $a^2b^2 + 2abc + c^2$.
3. $4x^2 - 4xy + y^2$.
4. $9x^2 - 16y^4$.
5. $x^6 - y^6$.
6. $64 - 16xy + y^2$.
7. $x^8 - y^6$.
8. $4m^2 - 4mn - n^2$.
9. $-9x^2 - 24xy + 16y^2$.
10. $a^{10}x^2 - 9y^4$.
11. $x^4 - y^9$.
12. $8x^2 - 16y^2$.

SECTION XXXII.

FACTORING.—ADDITIONAL THEOREMS.

REMARK.—Quantities with exponents, as a^3 or x^6, are called **powers.** a^3 is the *third power* of a, and x^6 is the sixth power of x.

84. IT is sometimes necessary to factor expressions like $a^5 - x^5$, and $a^5 + x^5$.

If we divide $a^5 - x^5$, or $a^8 - x^8$, by $a - x$, we find it divides exactly in each case. It will be the same with any other powers of a and x, if they are alike and are connected by the sign $-$.

If we divide $a^5 + x^5$ or $a^7 + x^7$, or the *sum* of any like *odd* powers of a and x, by $a + x$, it will be contained exactly.

These principles may be expressed thus:

Theorem IV.*—The difference of two quantities is an exact divisor of the difference of any like powers of those quantities.*

Theorem V.*—The sum of two quantities is an exact divisor of the sum of any like* odd *powers of those quantities.*

These theorems give us only one of the factors. The other must be found by actual division.

Expressions like $a^2 + x^2$ and $a^4 + x^4$, which are the sums of second and fourth powers, have no divisors.

Factor the following by finding one factor by Theorem IV. or Theorem V., and the other by division:

WRITTEN EXAMPLES.

1. $a^3 + b^3$.
2. $a^3 - b^3$.
3. $x^5 + y^5$.
4. $x^7 - y^7$.
5. $a^5b^5 + c^5$.
6. $x^3 + y^3$.
7. $c^3d^3 - x^3$.
8. $1 + x^3$.
9. $a^5 - 1$.
10. $1 - b^3$.

SECTION XXXIII.

GENERAL METHOD OF FACTORING.

IT will be noticed that two methods have been followed in the different examples already factored. In such expressions as $a^2 - 2ax$ and $4y + 4x$, or $a^3 - x^3$ and $a^5 + x^5$, we looked first for a divisor, which would be one factor, and then found the other factor by division. But in expressions like $a^2 + 2ab + b^2$ and $x^2 - 4$, we tried to find, at once, what two quantities would give them as their product. The student must bear in mind both methods, and use whichever the particular case seems to require.

Only two factors of any expression have been found. Expressions may have more than two factors. These are generally found by separating the quantity into two factors first, and then one or both of these into two others, and so on till none of them will factor.

Ex. Factor $a^3 - 2a^2x + ax^2$.

SOLUTION. We look first for a *monomial* factor, and find a; we divide by a and obtain the other factor, $a^2 - 2ax + x^2$. This last factor can be separated, according to Theorem II., into $a - x$ and $a - x$. Hence, the three factors are a, $a - x$, and $a - x$.

The work would be done thus:

$$\begin{aligned} a^3 - 2a^2x + ax^2 &= a(a^2 - 2ax + x^2), \\ &= a(a - x)(a - x). \end{aligned}$$

Ex. Factor $a^6 - x^6$.

SOLUTION. This is the difference of two squares; it will therefore factor, according to Theorem III., into $a^3 + x^3$ and $a^3 - x^3$. $a^3 + x^3$, according to Theorem V., can be divided by $a + x$. The other factor, found by division, is $a^2 - ax + x^2$. This cannot be factored. $a^3 - x^3$, according to Theorem IV., can be divided by $a - x$. The other factor, found by division, is $a^2 + ax + x^2$. This cannot be factored. Therefore the factors are $a + x$, $a^2 - ax + x^2$, $a - x$, and $a^2 + ax + x^2$.

The work would be performed thus:

$$\begin{aligned} a^6 - x^6 &= (a^3 + x^3)(a^3 - x^3), \\ &= (a + x)(a^2 - ax + x^2)(a - x)(a^2 + ax + x^2). \end{aligned}$$

Ex. Factor $3a^9 - 3ab^8$.

OPERATION.

$$\begin{aligned} 3a^9 - 3ab^8 &= 3a(a^8 - b^8) \\ &= 3a(a^4 + b^4)(a^4 - b^4) \\ &= 3a(a^4 + b^4)(a^2 + b^2)(a^2 - b^2) \\ &= 3a(a^4 + b^4)(a^2 + b^2)(a + b)(a - b) \end{aligned}$$

85. From the preceding explanations we derive, for factoring polynomials, the following general

Rule.—*I. Examine the quantity for a monomial factor, and if there is one, find by division the polynomial factor.*

II. Examine the polynomial to find—

1. *Whether it is a perfect square—to be factored by Theorem I. or II.*

2. *Whether it is the difference of two squares—to be factored by Theorem III.*

3. *Whether it is the difference of the like powers of two quantities—to be factored by Theorem IV.*

4. *Whether it is the sum of the like odd powers of two quantities—to be factored by Theorem V.*

III. Find whether any of the resulting polynomial factors can be further factored, and continue the operation till all the polynomial factors are prime quantities.

Factor the following:

WRITTEN EXAMPLES.

1. $a^3 - ab^2$.
2. $2x^3 - 4x^2y + 2xy^2$.
3. $3a^3 + 6a^2 + 3a$.
4. $ax^2y - axy^3$.
5. $5c^2d^2x + 10c^2dx^2 + 5c^2x^3$.
6. $x^2 - 4$.
7. $3x^4 - 3$.
8. $8x^2 - 8x + 2$.
9. $x^4 - 4x^2$.
10. $5x^6 - 20x^4$.
11. $x^3 + y^3$.
12. $x^6 - y^6$.
13. $4ax^7 - 4axy^6$.
14. $18a^3 + 24a^2b + 2ab^2$.
15. $40a^4 - 5a$.
16. $7a^{10} - 7a^2x^8$.
17. $a^5b^5xy - abx^5y^5$.
18. $3c^7 + 6c^5 + 3c^3$.
19. $4a^3 + 4a^2b + 4ab^2$.
20. $24 - 6x^4$.

SECTION XXXIV.

REVIEW.

MENTAL EXERCISE.

1. Multiply $-3ax$ by -2? Why is the sign of the product +? (Art. 40.)

2. Multiply x^2 by x^3. Why are the exponents added? (Art. 42.)

3. What are the values of $10 - (4 + 1)$ and $10 - 4 + 1$?

4. What difference does the parenthesis make?

5. What are the values of $(6 + 4) \div 2$ and $6 + 4 \div 2$?

Give the values of each of the following expressions:

6. $5 - 3 \times 2 + 1$.
7. $(5 - 3) \times 2 + 1$.
8. $(5 - 3)(2 + 1)$.
9. $(5 - 3)(3 + 1)$.
10. $6 - 5 - 8 + 10$.
11. $(6 - 5) - (3 + 10)$.
12. $(6 - 5) - 3 + 10$.
13. $6 - (5 - 3) + 10$.

14. What operation is indicated in Ex. 8?

15. Remove the parenthesis from $4 - (2 - x)$. Why are the signs of $2 - x$ changed? (Art. 46.)

16. If $4x - 3y = 12$,
and $y = 4$, how may the value of x be found?

17. If $3x + y = 17$,
$2x - 3y = 4$, how can we eliminate x? (Art. 59.)

18. From the same equations how can we eliminate y?

19. Divide x^7 by x^4. Why are the exponents subtracted? (Art. 67.)

20. Arrange with reference to a the qnantities $a^2 + b^2 + 2ab$ and $a^3 + b^3 + 3a^2b + 3ab^2$.

21. In performing division of polynomials, what is the first operation?

22. How are the successive terms of the quotient found?

23. Why does the quotient thus found show how many times the dividend contains the divisor? (Art. 71.)

24. $(x + y)(x + y)$ is the square of the sum of what two quantities?

25. Why is it the *square* of their sum? How else may it be expressed?

26. $(c + d)(c - d)$ is the product of the sum and difference of what two quantities?

27. What are the factors of a quantity?

28. If a quantity has two factors, and one of them is known, how can the other be found?

29. Give one factor each of $x^3 - y^3$ and $x^5 + y^5$.

30. Has $x^4 + y^4$ any factor?

31. Quantities which are perfect squares are factored by the aid of what theorems?

32. How can we tell whether a quantity is a perfect square?

PART II.

LEAST COMMON MULTIPLE.—FRACTIONS.—SIMPLE EQUATIONS.

SECTION XXXV.

LEAST COMMON MULTIPLE OF MONOMIALS.

86. A Multiple of a quantity is any number of times of that quantity. Hence, it is also any quantity that the first will exactly divide.

A Common Multiple of two or more quantities is any quantity that each will exactly divide.

87. The **Least Common Multiple** of two or more **numbers** is the least number that each will exactly divide.

The **Least Common Multiple** of two or more **algebraic quantities** is that multiple of the quantities which contains the least number of factors.

Thus, a^2x^2 is the least common multiple of a^2x and ax^2.

Note.—In Arithmetic the least common multiple is the least or smallest divisible number, but in Algebra we do not know that it is always the least quantity, since we do not know the values of the letters. But it is least as regards its coefficients and exponents, or number of factors.

88. What is the quantity of fewest letters that ax and x^2y will each exactly divide?

Solution. For a quantity to be divided exactly by ax it must contain the letters a and x. For the quantity to be divided exactly by x^2y it must contain 2 x's, or x^2, and y. Hence, to be divided exactly by both ax and x^2y the quantity must contain a, x^2, and y. From this we derive the following

***Rule.**—The least common multiple of two or more monomials composed of letters, consists of all the letters in those quantities, each with an exponent equal to the highest exponent of that letter in any quantity.*

ILLUSTRATION. The least common multiple
of c^2d^3x and ax^3y is $ac^2d^3x^3y$;
of abc^3m^2, a^3c^2m, and b^2cm is $a^3b^2c^3m^2$.

Give the least common multiple of each of the following sets of monomials.

MENTAL EXERCISE.

1. a^2, ab, and b^2.
2. xy, x^2y, and yz.
3. ab^2, bx^3, and xy.
4. m^3x, n^2x^3, and y^3.
5. ac^2, a^2d, and d^2x^3.
6. a, x^2y, and m^2n.
7. b^3y, a^2xy^3, and a^2y^3.
8. m^2ny^3, ay^2z^2, and an^2y^4.
9. a^2d, ad^2, d^3x, and x^3y^3.
10. cy^2, a^2, and xz^3.

89. If the monomials have coefficients, the least common multiple of these must be found as in Arithmetic. This can often be done mentally.

Give the least common multiple of each of the following sets of monomials.

MENTAL EXERCISE.

1. 4, 3, and 2.
2. 6, 2, and 3.
3. 5, 10, and 3.
4. 12, 3, and 4.
5. 3, 4, and 6.
6. 8, 3, and 2.
7. $14a$, $7ax$, and $2x^2y$.
8. $10x^3$, $4x^2$, and $5ax^3$.
9. $9xy$, $6ax^2$, and $2xy^3$.
10. $3ay$, a^3x, and $2ax^2$.
11. $4xy^3$, $2x^2$, and $3x^2y^3$.
12. $12ay^3$, $6a^2y$, and $2a^4$.

SECTION XXXVI.

LEAST COMMON MULTIPLE OF POLYNOMIALS.

90. WHERE the factors are not all letters or numbers, as in $a(a+x)$ and $(a-x)(a+x)$, the same principles apply as in the preceding examples.

***Rule.**—Take each factor the greatest number of times that it occurs in any quantity.*

ILLUSTRATION. The least common multiple of

$$a(a+x) \text{ and } (a-x)(a+x) \text{ is } a(a-x)(a+x),$$

and of

$$x^2(x-y)^2 \text{ and } ax(x-y)(x+y) \text{ is } ax^2(x-y)^2(x+y).$$

Give the factored form of the least common multiple of each of the following sets of factored expressions:

MENTAL EXERCISE.

1. $a(a+x)$ and $b(a+x)$.
2. $a(x-y)$ and $ax(x+y)$.
3. $4(a-b)$ and $(a-b)(a+b)$.
4. $a(a-x)^2$ and $x(a-x)^3$.
5. $3(x+4)$ and $(x+4)(x-5)$.
6. $2(x+6)$ and $4(x+6)^2$.
7. $5(a-y)(a+y)$ and $2(a+y)^3$.
8. $6(x+y)^2$ and $3a(x-y)(x+y)$.
9. $x(x+1)$, $x(x+1)(x+2)$, and $x(x+2)$.
10. $(a+b)(a+c)$, $(a+c)(c+b)$, and $(a+b)(c+b)$.

91. If the quantities are not in the factored form, as a^2-x^2 and $a^2+2ax+x^2$, they must be resolved into their factors, and the operation completed as before. The factors of the least common multiple must usually be actually multiplied together.

***Rule.**—Resolve the quantities into their prime factors, and take each factor the greatest number of times that it occurs in any quantity.*

Ex. What is the least common multiple of $a^2 - x^2$ and $a^2 + 2ax + x^2$?

SOLUTION. Factoring, we have $a^2 - x^2 = (a + x)(a - x)$,

and $a^2 + 2ax + x^2 = (a + x)^2$.

Taking $a - x$ once and $a + x$ twice, we have for the least common multiple,

$$(a - x)(a + x)^2 = a^3 + a^2x - ax^2 - x^3.$$

92. If the quantities have no common factor, as, a and bx or $(x + y)(x - y)$ and $a(a + b)$, the least common multiple will be the product of all the quantities. Thus, the least common multiple of $x + y$ and $x - y$ is $x^2 - y^2$.

Find the least common multiple of each of the following sets of quantities

WRITTEN EXAMPLES.

1. $3a + 3x$, $6a - 6x$, and $a^2 - x^2$.
2. $2y^2$, $x^2 - y^2$, $4x^2 - 4xy$, and $2x + 2y$.
3. $x^3 - y^3$, $x + y$, $x - y$, and $x^2 - y^2$.
4. c^3d, $c^2 - cd^2$, and $cx - dx$.
5. $a + b$, $(a + b)^2$, $a^2 - b^2$, and $(a - b)^2$.
6. $a^2 + 2ax + x^2$, $a^2 - 2ax + x^2$, and $a^3 - x^3$.
7. $5b^3c$, $10b^2 - 5by$, $2b - y$, and $4b^2 - y^2$.
8. $4a^2b$, $4a^2 - 2ab$, and $2a - b$.
9. $6xy^3$, $9xy$, $3a^2 - 3b^2$, $a - b$, and $a^2 + 2ab + b^2$.
10. $8c^2$, $4d^3$, $2c^3$, $4c^2d + 2cd^2$, and $c + d$.
11. $x + y$, $x - y$, $x^2 + y^2$.
12. mn, $ax + ay$, and xy.

MENTAL EXERCISE.

1. $a + b$ and $a - b$.
2. $x - y$, $x + y$, and $x^2 - y^2$.
3. a^2, $a + b$, and $a^2 + 2ab + b^2$.
4. $3x$, $x - y$, and $x^2 - 2xy + y^2$.
5. $c + d$ and $c^2 - d^2$.
6. a, $a - b$, and $a^3 - b^3$.
7. $x + y$, $x - y$, and $x^2 + y^2$.
8. $a - b$, $a + b$, $a^2 - b^2$, and $a^2 + b^2$.
9. $4x$, $8xy$, $a^2 - b^2$, and $a + b$.

SECTION XXXVII.

FRACTIONS.—GENERAL PRINCIPLES AND DEFINITIONS.

93. PRINCIPLE. An **Algebraic Fraction** may be regarded as a case of indicated division. Thus, $\frac{a}{b}$ indicates a divided by b, and is so read. The **numerator** is the **dividend**, the **denominator** is the **divisor**, and the **value** of the fraction is the **quotient**. Thus, $\frac{6}{2}$ indicates 6 divided by 2, and the quotient 3 is the value of the fraction. This is also shown as follows:

$$\begin{array}{ll} \text{Numerator or dividend,} & 8ab \\ \hline \text{Denominator or divisor,} & 2b \end{array} = 4a, \text{ value or quotient.}$$

An **Entire Quantity** is one that has no fractional parts, as ax^2 and $x + y$.

A **Mixed Quantity** is one that has both entire and fractional parts.

Reduction of Fractions is the process of changing the form of fractions without changing their value.

The rules for working algebraic fractions are similar to those for arithmetical ones. If the student is not perfectly familiar with these let him look over that part of his Arithmetic.

SECTION XXXVIII.

REDUCTION OF FRACTIONS TO THEIR LOWEST TERMS.

94. A FRACTION is said to be reduced to its lowest terms when its numerator and denominator contain the least possible number of factors.

PRINCIPLE. $\frac{ac}{bc} = \frac{a}{b}$, because to divide both dividend and divisor by the same quantity does not change the quotient. Therefore, to divide both numerator and denominator of a fraction by the same quantity, as c, does not change its value.

Ex. Reduce $\frac{6axy}{3ayz}$ to its lowest terms.

SOLUTION. We observe that both numerator and denominator of this fraction can be divided by 3, a, and y.

Dividing numerator and denominator by $3ay$, we find

$$\frac{6axy}{3ayz} = \frac{2x}{z}.$$

As these cannot be divided by any other quantity, the fraction is in its lowest terms.

The work is most easily performed by observing what factors occur in both numerator and denominator, and cancelling them. The remaining factors make the numerator and denominator of the fraction.

Reduce the following fractions to the lowest terms:

MENTAL EXERCISE.

1. $\frac{8acm}{16cmn}$.
2. $\frac{3mxy}{4amz}$.
3. $\frac{7b}{21abcx}$.
4. $\frac{4mnx}{8axy}$.
5. $\frac{12cdx}{6bcxy}$.
6. $\frac{5bcnz}{7cxy}$.
7. $\frac{x(a+b)}{(a+b)(a-b)}$.
8. $\frac{x(a+y)(a-x)}{xy(a-y)(a-x)}$.
9. $\frac{2(3x+2)(5x-1)}{4(5x-1)(4x-3)}$.
10. $\frac{4ax(2a-b)(2a+b)}{4xy(2a+b)(2a-b)}$.

95. Reduce $\frac{12a^2b^2x}{8ab^3y}$ to its lowest terms.

SOLUTION. We observe here that $4ab^2$ will divide both numerator and denominator. Dividing them by it, we find

$$\frac{12a^2b^2x}{8ab^3y} = \frac{3ax}{2by}.$$

The same result would have been attained more easily by cancelling out the common factors as before. This is really the same as dividing by one common factor at a time, instead of by the product of all together.

The work may be written thus:

$$\frac{\overset{3\ a}{\cancel{12}\,\cancel{a^2}\cancel{b^2}x}}{\underset{2\ b}{\cancel{8}\,\cancel{a}\cancel{b^3}y}} = \frac{3ax}{2by}.$$

Here we divide 4 into 12 and 8, obtaining 3 and 2; we divide a into a^2 and a, obtaining a and 1; and we divide b^2 into b^2 and b^3, obtaining 1 and b. The 1's are not expressed, except where there are no other factors to go with it, as

$$\frac{a}{2ax} = \frac{1}{2x}.$$

Rule.*—Cancel the numerical factors as in Arithmetic, and then cancel out the letters as far as can be done.*

Reduce the following to their lowest terms:

MENTAL EXERCISE.

1. $\dfrac{10c^2d}{15cd^3}$.
2. $\dfrac{6a^2b}{8ab^5}$.
3. $\dfrac{3mn^2x}{6m^2n^5x^3}$.
4. $\dfrac{7a^5bc^6xy^3}{4a^6bc^5x^2y}$.
5. $\dfrac{20a^7xy^3z}{10a^2bcx^3}$.
6. $\dfrac{14b^3c^2mn^3y^6}{28b^3cd^2x^3y^4}$.
7. $\dfrac{12a^2bc^3dm^4n}{18a^5b^2c^3dx^3y^2}$.
8. $\dfrac{4b^2c^2mx^4y}{9b^2c^2mx^4y}$.
9. $\dfrac{15m^3nx^4z}{25m^2n}$.
10. $\dfrac{6b^2c^3dx^2}{12b^3c^4dx^2}$.

96. Reduce $\dfrac{a^3 - ax^2}{a^3 + 2a^2x + ax^2}$ to its lowest terms.

Solution. We observe that a will divide both terms of the fraction.

Dividing, we obtain $\dfrac{a^2 - x^2}{a^2 + 2ax + x^2}$.

We will find next that $a + x$ will divide both terms.

Dividing, we have $\dfrac{a - x}{a + x}$.

The work is most easily performed by factoring both terms and cancelling out common factors. Thus,

$$\frac{a^3 - ax^2}{a^3 + 2a^2x + ax^2} = \frac{a(a^2 - x^2)}{a(a^2 + 2ax + x^2)} = \frac{a(a+x)(a-x)}{a(a+x)(a+x)} = \frac{a-x}{a+x}.$$

Rule.—I. Divide the numerator and denominator by such quantities as will divide both. Or,

II. Separate the numerator and denominator into their factors, and cancel those common to both.

ILLUSTRATION.

$$\frac{x^2y - xy^2}{x^3y - xy^3} = \frac{xy(x-y)}{xy(x^2 - y^2)} = \frac{xy(x-y)}{xy(x-y)(x+y)} = \frac{1}{x+y}.$$

Reduce the following to their lowest terms:

WRITTEN EXAMPLES.

1. $\frac{x - y}{x^2 - y^2}$.
2. $\frac{x^2 - y^2}{x^2 + 2xy + y^2}$.
3. $\frac{ab}{a^2 + ab}$.
4. $\frac{ax - ay}{ax^2 - ay^2}$.
5. $\frac{4x + 4y}{5x + 5y}$.
6. $\frac{7x - 7y}{x^2 - y^2}$.
7. $\frac{x^2 + 2xy + y^2}{x^2 - y^2}$.
8. $\frac{a^3 - 2a^2b + ab^2}{a^2b - b^3}$.
9. $\frac{10a^3b + 15ab^3 - 20a^2bc^3}{30ab^3 + 25abc^3 + 10a^2bx^2}$.
10. $\frac{12ax^3y^4z^2 - 28ax^5y^3z^2}{32x^6yz^3 + 8x^4y^4z^4}$.
11. $\frac{x^3 - y^3}{x^2 - 2xy + y^2}$.
12. $\frac{x^2 - y^2}{x^3 + y^3}$.
13. $\frac{a^3 - ab^2}{a^3 + 2a^2b + ab^2}$.
14. $\frac{4a^3 - 4ax^2}{6a^3 - 12a^2x + 6ax^2}$.

MENTAL EXERCISE.

1. $\dfrac{a}{ax + ay}$.
2. $\dfrac{mn}{m^3 - mn^2}$.
3. $\dfrac{c - d}{c^2 - d^2}$.
4. $\dfrac{a^2 - b^2}{a^2 - 2ab + b^2}$.
5. $\dfrac{a^2 + 2ax + x^2}{a^2 - x^2}$.
6. $\dfrac{3m + 3n}{8m + 8n}$.
7. $\dfrac{a^2 + ab + ac}{ax + ay + az}$.
8. $\dfrac{2x - 2y}{x^2 - 2xy + y^2}$.
9. $\dfrac{c^2 - d^2}{c^2 - cd}$.
10. $\dfrac{x^2 + xy}{x^2 + 2xy + y^2}$.
11. $\dfrac{am + bm}{ax + bx}$.
12. $\dfrac{a^2 + a}{ax + ay}$.

SECTION XXXIX.

REDUCTION OF FRACTIONS TO WHOLE OR MIXED QUANTITIES.

97. PRINCIPLE. $\frac{ac}{a} = c$ and $\frac{ac + b}{a} = c + \frac{b}{a}$, because the value of a fraction is the quotient resulting from the indicated division, and will be equal to the actual quotient, if such can be obtained.

***Rule.**—Divide the denominator into the numerator.*

NOTE.—If the denominator be a monomial, divide it into each term of the numerator which it will divide exactly, and write it under the others. Reduce the fractional parts to their lowest terms.

Ex. Reduce $\frac{a^2 + ax + ay}{a}$ to an entire quantity.

OPERATION.

$$\frac{a^2 + ax + ay}{a} = a + x + y.$$

In the same way $\dfrac{c^2 + cd + d^2}{c^2} = 1 + \dfrac{d}{c} + \dfrac{d^2}{c^2}$,

and $\dfrac{ay^2 + axy + xy^2}{xy} = \dfrac{ay}{x} + a + y$.

Note.—If the denominator be a polynomial, perform the indicated division as in Long Division.

Ex. Reduce $\dfrac{a^3 - x^3}{a + x}$.

OPERATION.

$$\begin{array}{rl}
a+x\,) & a^3 - x^3 \quad (a^2 - ax + x^2 - \dfrac{2x^3}{a+x} \\
& \underline{a^3 + a^2x} \\
& \quad - a^2x - x^3 \\
& \quad \underline{- a^2x - ax^2} \\
& \qquad\qquad ax^2 - x^3 \\
& \qquad\qquad \underline{ax^2 + x^3} \\
& \qquad\qquad\qquad - 2x^3
\end{array}$$

Reduce the following fractions to whole or mixed quantities:

MENTAL EXERCISE.

1. $\dfrac{x^2 + xy}{x}$.
2. $\dfrac{a^2 + ab + b^2}{a}$.
3. $\dfrac{4c^3 + 6c^2d + 8cd^2}{2c^2}$.
4. $\dfrac{xy + y^2}{y^2}$.
5. $\dfrac{c^2 - cd}{c^2}$.
6. $\dfrac{x^3 + x^2 + x}{x^2}$.
7. $\dfrac{5a^2y + 7ay^2 + ay^3}{ay}$.
8. $\dfrac{x^2y + xy^2 + y^3}{y^2}$.
9. $\dfrac{9a^2cd + 18a^2c^2d + 12acd^2}{3acd}$.
10. $\dfrac{a^2 + ab + c^2}{a^2}$.

WRITTEN PROBLEMS.

1. $\dfrac{x^3 + 4x^2y + 3xy^2}{x + y}$.
2. $\dfrac{3a^2b - 3ab^2}{a^2 - b^2}$.
3. $\dfrac{6c^2 - 8d^2}{3c - 2d}$.
4. $\dfrac{x^5 - y^5}{x - y}$.
5. $\dfrac{a^6 + b^6}{a + b}$.
6. $\dfrac{x^3 - y^3}{x + y}$.
7. $\dfrac{c^3 - d^3}{c^2 - cd + d^2}$.
8. $\dfrac{a^4 - 7a^3b + 5a^2b^2}{a^2 - 8b^2}$.
9. $\dfrac{x^2 + xy - y^2}{x + y}$.
10. $\dfrac{x^2 - xy + y^2}{x - y}$.

SECTION XL.

REDUCTION OF ENTIRE AND MIXED QUANTITIES TO FRACTIONS.

98. Principles. These operations are exactly the reverse of those in the last Section.

REDUCTION OF ENTIRE QUANTITIES TO FRACTIONS HAVING GIVEN DENOMINATORS.

***Rule.**—To find the numerator, multiply the entire quantity by the given denominator.*

Illustration. $4a = \dfrac{4ab}{b}$, and $5ax = \dfrac{10ax^2y}{2xy}$.

MENTAL EXERCISE.

1. Reduce $5x^2$ to a fraction whose denominator is xy.
2. Reduce $a + b$ to a fraction whose denominator is $2ab$.
3. Reduce $4x^2y$ to a fraction whose denominator is $x - y$.

4. Reduce $x - y$ to a fraction whose denominator is $x + y$.

5. Reduce $a + b$ to a fraction whose denominator is $a + b$.

6. Reduce $m^2 + 2mn + n^2$ to a fraction whose denominator is $4mn$.

REDUCTION OF MIXED QUANTITIES TO FRACTIONS.

99. ***Rule.****—Multiply the entire quantity by the denominator, and add or subtract the numerator, according as the sign before the fraction is + or −. Write the result over the denominator.*

Illustration.

$$a + \frac{x^2}{a} = \frac{a^2 + x^2}{a} \quad \text{and} \quad a + b - \frac{c^2}{a - b} = \frac{a^2 - b^2 - c^2}{a - b}.$$

Reduce the following mixed quantities to fractions:

MENTAL EXERCISE.

1. $c + \dfrac{ab}{x}$.
2. $mn + \dfrac{n^2}{m}$.
3. $a + b + \dfrac{c^2}{a}$.
4. $a - \dfrac{x^4}{a}$.
5. $b - c - \dfrac{y^2}{x}$.
6. $x + \dfrac{x^2 + y^2}{a}$.
7. $x + \dfrac{3x}{4}$.
8. $2y^2 - \dfrac{4y^2}{5}$.
9. $a + \dfrac{x^2}{a + b}$.
10. $1 + \dfrac{b^2}{a^2}$.

100. Reduce $2x + y - \dfrac{x^2 + y^2}{x - 3y}$ to a fraction.

First Solution.

$$\begin{array}{ll} \text{Multiplying} & 2x + \quad y \\ \text{by} & \underline{\;x - \quad 3y\;} \\ & 2x^2 + \quad xy \\ & \underline{\qquad - 6xy - 3y^2} \\ \text{we have} & 2x^2 - 5xy - 3y^2 \\ \text{Subtracting the numerator,} & \underline{\;x^2 \qquad\quad + \; y^2} \\ \text{we have} & x^2 - 5xy - 4y^2 \end{array}$$

Therefore, $2x + y - \dfrac{x^2 + y^2}{x - 3y} = \dfrac{x^2 - 5xy - 4y^2}{x - 3y}$.

In this solution the operations called for in the rule are directly performed.

Second Solution.

$$2x + y - \frac{x^2 + y^2}{x - 3y} = \frac{2x^2 - 5xy - 3y^2 - x^2 - y^2}{x - 3y},$$

$$\frac{2x^2 - 5xy - 3y^2 - x^2 - y^2}{x - 3y} = \frac{x^2 - 5xy - 4y^2}{x - 3y}.$$

In the second solution, after multiplying the entire quantity by the denominator, we subtract the numerator by *adding it with all its signs changed.* This second method is generally the better one to use when the denominator is small, or when the multiplication can be performed mentally.

Illustration. $a + b - \dfrac{c^2 + d^2}{a} = \dfrac{a^2 + ab - c^2 - d^2}{a}$, and

$$m - n - \frac{a - b}{m + n} = \frac{m^2 - n^2 - a + b}{m + n}.$$

Reduce the following mixed quantities to fractions:

WRITTEN EXAMPLES.

1. $a + b + \dfrac{a^2 + b^2}{a}$.

2. $2x + y - \dfrac{3y^2 - 2xy}{x}$.

3. $a + b - \dfrac{a^2 + b^2}{a + b}$.

4. $3c - 4d + \dfrac{5c^2 - 3d^2}{3c}$.

5. $2x - 3y - \dfrac{4a^2 + x^2}{2a + x}$.

6. $a + b - \dfrac{c + d}{x + y}$.

7. $a - b - \dfrac{a^2 - b^2}{a + b}$.

8. $a^2 - ab + b^2 - \dfrac{a^3 - 2b^3}{a - b}$.

MENTAL EXERCISE.

1. $m - n + \dfrac{n^2}{m + n}$.

2. $a + b - \dfrac{a^2}{a - b}$.

3. $x - y - \dfrac{2xy}{x - y}$.

4. $a + b - \dfrac{a^2 + b^2}{a + b}$.

5. $c + 2d + \dfrac{3c - 4d}{5}$.

6. $2a + 4b - \dfrac{3a - 7b}{3}$.

7. $1 - \dfrac{x}{x + y}$.

8. $\dfrac{a}{a - b} - 1$.

9. $1 + \dfrac{n}{m - n}$.

10. $\dfrac{x - y}{y} + 1$.

SECTION XLI.

REDUCTION OF FRACTIONS TO EQUIVALENT ONES HAVING THE LEAST COMMON DENOMINATOR.

101. Principle. $\dfrac{a}{b} = \dfrac{ac}{bc}$, because multiplying both dividend and divisor by the same quantity does not change the quotient, and therefore multiplying numerator and denominator of a fraction by the same quantity does not change the value of the fraction.

Ex. Reduce $\dfrac{a}{b}$, $\dfrac{b}{d}$, and $\dfrac{x}{b^2}$ to equivalent fractions having the least common denominator.

Solution. In order to make the denominators of the fractions the same, we must find the proper multipliers for each fraction. This may be done by finding the least common multiple of the denominators, and then dividing each denominator into it. This will give the multiplier for each fraction.

The least common multiple of the denominators is b^2d. Dividing this by the first denominator gives bd for the first multiplier; dividing by the second denominator gives b^2 for the second; and dividing by the third gives d for the other multiplier. Now, multiplying by these, we have

$$\frac{a \times bd}{b \times bd} = \frac{abd}{b^2d}, \quad \frac{b \times b^2}{d \times b^2} = \frac{b^3}{b^2d}, \text{ and } \frac{x \times d}{b^2 \times d} = \frac{dx}{b^2d}.$$

These fractions are equal to the first ones, and have a common denominator, b^2d, which is the least, and is the same as the least common multiple before found.

Rule.*—Find the least common multiple of the denominators; this will be the least common denominator. Divide the denominator of each fraction into this, and multiply the result by the numerator; this will give the new numerator of each fraction.*

Ex. Reduce $\frac{a}{b^2}$, $\frac{c}{bc}$, and $\frac{1}{cx}$ to equivalent fractions having the least common denominator.

Solution. We observe that b^2cx is the least common multiple of the denominators. It is therefore the least common denominator. We write the work thus:

$$\frac{a}{b^2} = \frac{\quad}{b^2cx}, \quad \frac{c}{bc} = \frac{\quad}{b^2cx}, \quad \frac{1}{cx} = \frac{\quad}{b^2cx},$$

and then proceed to find the new numerators.

We divide the first denominator, b^2, into the common denominator, b^2cx. It is contained cx times. We multiply this by the old numerator, a, and have acx for the new numerator.

In the same way we find the other numerators to be bcx and b^2; therefore,

$$\frac{a}{b^2}=\frac{acx}{b^2cx},\quad \frac{c}{bc}=\frac{bcx}{b^2cx},\quad \frac{1}{cx}=\frac{b^2}{b^2cx}.$$

Ex. Reduce $\frac{x}{x-y}$, $\frac{y}{x+y}$, and $\frac{x^2}{x^2-y^2}$ to equivalent fractions having the least common denominator.

Solution. We observe that the least common multiple of the denominators is x^2-y^2.

We write the work thus:

$$\frac{x}{x-y}=\frac{\quad}{x^2-y^2},\quad \frac{y}{x+y}=\frac{\quad}{x^2-y^2},\quad \frac{x^2}{x^2-y^2}=\frac{\quad}{x^2-y^2}.$$

Dividing each old denominator into the common denominator, and multiplying the result by the old numerator to get the new numerators, the result is

$$\frac{x}{x-y}=\frac{x^2+xy}{x^2-y^2},\quad \frac{y}{x+y}=\frac{xy-y^2}{x^2-y^2},\quad \frac{x^2}{x^2-y^2}=\frac{x^2}{x^2-y^2}.$$

If the least common multiple of the denominators cannot be easily determined by inspection, find it as directed in Art. 91.

Reduce each of the following sets of fractions to equivalent ones having the least common denominator

WRITTEN EXAMPLES.

1. $\frac{x}{3}$, $\frac{y}{x}$, and $\frac{a}{x^2}$.

2. $\frac{a}{b+c}$, and $\frac{b}{b-c}$.

3. $\frac{a}{c+d}$, $\frac{b}{c-d}$, and $\frac{c^2}{c^2-d^2}$.

4. $\frac{m}{n}$, $\frac{am}{n^2}$, $\frac{x}{m+n}$, and $\frac{y}{m-n}$.

5. $\frac{a}{3}$, $\frac{b}{6}$, $\frac{c}{8}$, and $\frac{a}{10}$.

6. $\frac{a}{6(a-b)}$, $\frac{b^2}{(a-b)^2}$, and $\frac{a^2b}{(a-b)^3}$.

7. $\frac{c}{5}$, $\frac{a}{7}$, $\frac{b}{a+b}$, and $\frac{a^2}{a^2+2ab+b^2}$.

8. $\frac{2}{x}$, $\frac{xy}{x^2+xy}$, and $\frac{y^2}{x^2-y^2}$.

9. $\frac{a^2}{a^2-2ab+b^2}$, $\frac{ab}{a^2+2ab+b^2}$, $\frac{b^2}{a^2-b^2}$.

10. $\frac{4}{x^3}$, $\frac{5}{x^2-y^2}$, and $\frac{8}{x-y}$.

MENTAL EXERCISE.

If the denominators have no common factor, their product will be the least common denominator, and the new numerator may be found by multiplying each numerator by all the denominators except its own.

Note.—If any of the quantities are entire, their denominators may be regarded as 1.

1. $\frac{a}{b}$, $\frac{b}{c}$, and $\frac{c}{d}$.

2. $\frac{a}{x}$, $\frac{b}{x^2}$, and $\frac{c}{x^3}$.

3. $\frac{1}{a+b}$ and $\frac{1}{a-b}$.

4. $\frac{cx}{ay^2}$, $\frac{cy}{ax^2}$, and $\frac{xy}{a}$.

5. $\frac{x^2}{x^2-y^2}$, $\frac{1}{x+y}$, and $\frac{1}{x-y}$.

6. $\frac{c-d}{c+d}$ and $\frac{c^2}{c^2-d^2}$.

7. 4 and $\frac{x}{y}$.

8. $\frac{6}{x^2-2xy+y^2}$, and $\frac{2}{x-y}$.

9. $\frac{a-x}{a+x}$ and $\frac{a+x}{a-x}$.

10. $\frac{2}{y^2}$ and $\frac{a}{x+y}$.

SECTION XLII.

ADDITION OF FRACTIONS.

102. Principle. $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$, because the number of times that a contains c, plus the number of times that b contains c, must be equal to the number of times that a and b together contain c.

***Rule.**—Reduce the fractions to the least common denominator; add the numerators, and write the sum over the common denominator.*

Ex. Collect $\frac{a}{3}+\frac{b}{c}+\frac{d}{c^2}$.

OPERATION.

$$\frac{a}{3}+\frac{b}{c}+\frac{d}{c^2}=\frac{ac^2}{3c^2}+\frac{3bc}{3c^2}+\frac{3d}{3c^2}=\frac{ac^2+3bc+3d}{3c^2}.$$

Ex. Collect $\frac{2}{x+y}+\frac{3}{x-y}+\frac{2x}{x^2-y^2}$.

OPERATION.

$$\frac{2}{x+y}+\frac{3}{x-y}+\frac{2x}{x^2-y^2}=\frac{2x-2y}{x^2-y^2}+\frac{3x+3y}{x^2-y^2}+\frac{2x}{x^2-y^2}$$

$$=\frac{7x+y}{x^2-y^2}.$$

Collect the following expressions:

WRITTEN EXAMPLES.

1. $\frac{1}{2}+\frac{2}{3}+\frac{5}{12}$.

2. $\frac{x}{2}+\frac{5x}{6}+\frac{3x}{8}$.

3. $\frac{a}{2}+\frac{3}{b}+\frac{c}{d}$.

4. $\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x^3}$.

5. $\frac{a+b}{3}+\frac{a-b}{6}+\frac{a+b}{12}$.

6. $\frac{x+y}{x-y}+\frac{x-y}{x+y}$.

7. $\frac{c+d}{c-d}+\frac{c}{d}$.

8. $\frac{1}{x+y}+\frac{1}{x-y}+\frac{x}{x^2-y^2}$.

9. $\frac{x}{x+y}+\frac{y^2}{(x+y)^2}+\frac{x^3}{(x+y)^3}$.

10. $\frac{a}{x+y}+\frac{b^2}{x^2-y^2}+\frac{c}{x}$.

11. $\frac{a}{a-4}+\frac{2}{a+2}$.

12. $\frac{c}{a^2b}+\frac{b}{c^2d}+\frac{d}{ac^2}$.

13. $\frac{n}{m-n}+\frac{m^2}{m^2-n^2}+\frac{n^2}{m^2}+\frac{n}{m}$.

14. $\frac{a^2}{a^3-b^3}+\frac{1}{a-b}$.

MENTAL EXERCISE.

1. $\frac{a}{x}+\frac{b}{y}$.

2. $\frac{c}{d}+\frac{b^2}{d^2}$.

3. $\frac{1}{x+y}+\frac{1}{x}$.

4. $\frac{1}{a-b}+\frac{1}{a+b}$.

5. $\frac{a+c}{2}+\frac{a+c}{6}$.

6. $\frac{1}{a-b}+\frac{a}{a^2-b^2}$.

7. $\frac{c}{ax}+\frac{d}{ay}+\frac{a}{xy}$.

8. $\frac{x}{x+y}+\frac{1}{z}$.

9. $\frac{1}{a+x}+\frac{1}{a-x}$.

10. $\frac{1}{x+1}+\frac{1}{x-1}$.

11. $\frac{x}{y}+\frac{y}{x}$.

12. $\frac{a+b}{a+b+c}+\frac{c}{a+b+c}$.

13. $\frac{x+y}{x^2}+\frac{x-y}{x^2}$.

14. $\frac{1+x}{x}+\frac{1-x}{y}$.

SECTION XLIII.

SUBTRACTION OF FRACTIONS.

103. PRINCIPLE. $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$, because the difference between the number of times that a contains c and the number of times that b contains c must be equal to the number of times that the difference between a and b contains c.

Rule.—*Reduce the fractions to the least common denominator; subtract the numerator of the subtrahend from that of the minuend, and write the remainder over the least common denominator.*

Perform the following indicated subtractions:

WRITTEN EXAMPLES.

1. $\frac{2}{3} - \frac{3}{5}$.
2. $\frac{3x}{4} - \frac{3x}{8}$.
3. $\frac{a^2}{b} - \frac{b^2}{a}$.
4. $\frac{c+d}{c-d} - \frac{c-d}{c+d}$.
5. $\frac{x-y}{x^2+2xy+y^2} - \frac{1}{x+y}$.
6. $\frac{x+y}{x} - \frac{x-y}{y}$.
7. $\frac{3+a}{6} - \frac{4+b}{12}$.
8. $\frac{2y}{x-y} - \frac{3x}{x+y}$.
9. $\frac{x^2+y^2}{x^2-y^2} - \frac{x-y}{x+y}$.
10. $\frac{a^2+ab+b^2}{a^2-b^2} - \frac{a-b}{a+b}$.

MENTAL EXERCISE.

1. $\frac{1}{a} - \frac{1}{b}$.
2. $\frac{x}{y} - \frac{a}{b}$.
3. $\frac{1}{a} - \frac{1}{x+y}$.
4. $\frac{x+y}{3} - \frac{x-y}{3}$.
5. $\frac{c+d}{x} - \frac{x-y}{c}$.
6. $\frac{a}{x+y} - \frac{b}{x}$.

EXAMPLES CONTAINING BOTH ADDITION AND SUBTRACTION.

104. ***Rule.—Reduce the fractions to the least common denominator, and add or subtract the numerators according as the sign of the fraction is + or −.***

Collect the following expressions:

WRITTEN EXAMPLES.

1. $\dfrac{a+b}{a-b}+\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a-b}{a+b}.$

2. $\dfrac{x^2}{y^2}+\dfrac{x-y}{x}-\dfrac{x+y}{y}.$

3. $\dfrac{b^2}{c^2}-\dfrac{b+c}{b-c}-\dfrac{c^2}{b^2-c^2}.$

4. $\dfrac{5}{x+3}+\dfrac{3x}{x-4}-\dfrac{x}{5}.$

5. $\dfrac{x}{x-y}+\dfrac{2x}{x+y}-\dfrac{2xy}{x^2-y^2}.$

6. $\dfrac{1}{a-x}+\dfrac{1}{a+x}-\dfrac{2}{x}.$

7. $\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}-\dfrac{a^2+b^2}{a^2-b^2}.$

8. $\dfrac{2x}{2x-4}-\dfrac{x}{x+2}+\dfrac{1}{2}.$

9. $\dfrac{x}{x^2-2xy+y^2}+\dfrac{4}{x-y}-\dfrac{1}{x}.$

10. $\dfrac{a^2}{a^2-1}-\dfrac{a}{a+1}+\dfrac{a}{a-1}.$

11. $\dfrac{1}{2-x}-\dfrac{2}{2+x}+\dfrac{3-x}{4-x^2}.$

12. $\dfrac{2a-b}{x-a}+\dfrac{a-2b}{x+a}+\dfrac{3ab}{x^2-a^2}.$

SECTION XLIV.

ADDITION AND SUBTRACTION OF MIXED QUANTITIES.

105. WHERE mixed quantities are to be added or subtracted—1st. The entire parts may be added or subtracted separately; or, 2d. The quantities may be reduced to fractions, and then added or subtracted according to the preceding rules.

FIRST METHOD.

Ex. Subtract $2a-x+\dfrac{3a}{b}$ from $4a-3x-\dfrac{8a}{b}$.

Solution. Since in subtraction we change the signs of the subtrahend and add it to the minuend, we will here change all the signs of $2a - x + \frac{3a}{b}$ and annex or add it to $4a - 3x - \frac{8a}{b}$.

Thus, $4a - 3x - \frac{8a}{b} - 2a + x - \frac{3a}{b}$, which, when collected, gives $2a - 2x - \frac{11a}{b}$.

***Rule.**—Connect the quantities by the proper signs, changing the signs of the subtrahend if one is to be subtracted, and collect the entire parts and the fractions separately.*

WRITTEN EXAMPLES.

1. Add $2xy + \frac{3x}{x+y}$ and $xy - \frac{x}{x-y}$.

Remark.—Writing this out according to rule, we have

$$2xy + \frac{3x}{x+y} + xy - \frac{x}{x-y}$$

2. Subtract $4c^2 + \frac{1}{a+b}$ from $5c^2 + \frac{1}{a-b}$.

3. Add $2x + 3y + \frac{x^2}{y}$ to $3z - 2y + \frac{a^2}{b}$.

4. Subtract $a + x - \frac{x-y}{4}$ from $2a - y - \frac{x+y}{6}$.

SECOND METHOD.

106. ***Rule.**—Reduce the mixed quantities to fractions, and add or subtract these according to the preceding rules.*

WRITTEN EXAMPLES.

1. Add $1 + \frac{y}{x}$ to $1 + \frac{x}{y}$.

2. Add $a + \frac{b^2}{a}$ to $a - \frac{a^2}{b}$.

3. Subtract $2x + \frac{3x}{4}$ from $5x - \frac{x}{6}$.

4. Subtract $3 - \frac{x - y}{x + y}$ from $2 + \frac{x - y}{x + y}$.

SECTION XLV.

MULTIPLICATION AND DIVISION OF FRACTIONS.

107. Principle I. $\frac{a}{b} \times c = \frac{ac}{b}$, because multiplying the dividend increases the quotient as many times, and therefore multiplying the numerator by c increases the value of the fraction, as many times.

To multiply a fraction by any quantity multiply the numerator by that quantity.

Principle II. $\frac{a}{b} \div d = \frac{a}{bd}$, because multiplying the divisor by any quantity similarly decreases the quotient, and therefore multiplying the denominator by d divides the value of the fraction by that quantity.

To divide a fraction by any quantity, multiply the denominator by that quantity.

Principle III. $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, because $\frac{a}{b} \times \frac{c}{d}$ means $\frac{a}{b}$ multiplied by c and *divided* by d.

To multiply one fraction by another, multiply the numerators together for a new numerator, and the denominators together for a new denominator.

Principle IV. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$, because $\frac{a}{b} \div \frac{c}{d}$ means $\frac{a}{b}$ divided by c and multiplied by d.

To divide one fraction by another, invert the divisor and multiply.

MULTIPLICATION OF FRACTIONS.

108. Ex. Multiply $\frac{ax}{by}$ by $\frac{b}{x}$.

Solution. According to Principle III. we will multiply the numerators together and the denominators together. Before the work is performed we can see that both numerator and denominator will contain b and x, which must be cancelled out before the fraction will be in its lowest terms. It will shorten the work to do the cancelling before the multiplication is performed.

The work would then be $\frac{ax}{by} \times \frac{b}{x} = \frac{a}{y}$.

From Principles I. and III. we have the following general

Rule.*—Multiply the numerators together for the new numerator, and the denominators for the new denominator, first cancelling out all factors common to any numerator and any denominator.*

Note.—In the following examples in multiplication and division of fractions mixed quantities must first be reduced to a fractional form.

Perform the following indicated multiplication:

MENTAL EXERCISE.

1. $\frac{x}{y} \times \frac{a}{b}$.
2. $\frac{bc}{d} \times \frac{a}{c}$.
3. $\frac{ax}{by} \times y$.
4. $\frac{a^2}{b^2} \times \frac{b}{a}$
5. $\frac{amx}{bny} \times \frac{bcy}{adx}$.
6. $\frac{ac^2}{xy} \times axy$.

7. $\frac{bcd}{axy} \times \frac{xdy}{bd}$.

8. $\frac{4}{x^3} \times \frac{y^2}{z}$.

9. $\frac{12cd^2}{b^2x} \times \frac{xy}{3cd^2}$.

10. $b^2d \times \frac{1}{ab^2}$.

11. $\frac{4ax}{15} \times 5$.

12. $\frac{7}{2x} \times 4xy$.

13. $\frac{5x}{4} \times 8$.

14. $\frac{9x}{2} \times 2$.

15. $\frac{4}{3x} \times 6x$.

WRITTEN EXAMPLES.

1. $\frac{a}{b^2} \times \frac{x+y}{a+b}$.

2. $\frac{x}{y} \times \frac{x-y}{x^2}$.

3. $\frac{a+b}{a-b} \times \frac{a-b}{a^2+2ab+b^2}$.

4. $\frac{c^2}{d^2} \times \frac{a^2-b^2}{c} \times \frac{d}{a+b}$.

5. $\frac{4-x^2}{6} \times \frac{9}{2+x}$.

6. $\frac{x^2-2xy+y^2}{x+y} \times \frac{x^2-y^2}{x} \times \frac{x^3}{x-y}$.

7. $\frac{3x+3y}{cd} \times \frac{ab}{4x+4y}$.

8. $\left(a - \frac{x^2}{a}\right) \times \frac{a^2}{a+x}$.

9. $\left(1 + \frac{x-y}{x+y}\right) \times \left(1 - \frac{y^2}{x^2}\right)$.

10. $\frac{a+x}{a^2+ax+x^2} \times \frac{a^3-x^3}{a^2-x^2} \times \frac{a+x}{a-x}$.

MENTAL EXERCISE.

1. $\dfrac{a}{x+y} \times \dfrac{x^2-y^2}{ab}$.

2. $\dfrac{a^2+2ab+b^2}{a-b} \times \dfrac{1}{a+b}$.

3. $\dfrac{ax+ay}{bx-by} \times \dfrac{b}{a}$.

4. $\dfrac{ax+ay}{bx-by} \times \dfrac{x-y}{x+y}$.

5. $\dfrac{3c+3d}{x} \times \dfrac{x^2}{c+d}$.

6. $\dfrac{c-d}{x^2} \times \dfrac{y^2}{c+d} \times \dfrac{c+d}{c-d}$.

7. $\dfrac{x^2-2xy+y^2}{x+y} \times \dfrac{1}{x-y}$.

8. $\dfrac{a^2-b^2}{b} \times \dfrac{1}{a+b}$.

9. $\dfrac{4-y^2}{6} \times \dfrac{3}{2-y}$.

10. $\dfrac{ax-ay}{a} \times \dfrac{c}{bx-by}$.

11. $\dfrac{4+x}{5} \times 10$.

12. $\dfrac{2x-6}{x} \times 4x$.

13. $\dfrac{3x}{2x+5} \times (2x+5)$.

14. $\dfrac{8x}{x+2} \times 2(x+2)$.

15. $\dfrac{1}{2x-1} \times (x-2)(2x-1)$.

16. $\dfrac{4}{3-x} \times (3-x)(2-x)$.

NOTE.—$\frac{3}{4}x$ is equivalent to $\frac{3}{4} \times x$, and equals $\dfrac{3x}{4}$.

Also, $\frac{2}{3}(x+5) = \dfrac{2x+10}{3}$, and $\frac{1}{2}(4x-6) = 2x-3$.

17. $\dfrac{3}{4}(2x-5)$.

18. $\dfrac{2}{3}\left(\dfrac{x+7}{2}\right)$.

19. $\dfrac{1}{4}(6x-10)$.

20. $\dfrac{1}{2}\left(\dfrac{4+8x}{3}\right)$.

21. $\dfrac{2}{5}\left(\dfrac{3x+4}{6}\right)$.

22. $\dfrac{4}{9}\left(\dfrac{6+3x}{8}\right)$.

DIVISION OF FRACTIONS.

109. From Principles II. and IV. we have the following general

***Rule.**—Invert the divisor, and proceed as in multiplication.*

NOTE.—Where the divisor is an entire quantity its denominator may be regarded as 1.

Perform the following indicated division:

WRITTEN EXAMPLES.

1. $\frac{a}{b} \div \frac{c}{d}$.

2. $\frac{a+b}{c+a} \div \frac{x}{y}$.

3. $\frac{x+y}{x-y} \div \frac{x+y}{x}$.

4. $\frac{x^2-y^2}{x^2+2xy+y^2} \div \frac{x-y}{x+y}$.

5. $\frac{a}{c} \times \frac{b}{a} \div \left(\frac{b}{c} \times \frac{x}{y}\right)$.

6. $\left(1+\frac{x}{a}\right) \div \left(1-\frac{x}{a}\right)$.

7. $\left(x-\frac{xy}{x+y}\right) \div \left(x+\frac{xy}{x-y}\right)$.

8. $\frac{a^3+x^3}{x^3} \div \frac{a+x}{xy}$.

9. $\frac{1}{x^2-y^2} \div \frac{1}{x-y}$.

10. $\frac{a+b}{a^2-b^2} \div \frac{a+b}{a-b}$.

11. $\frac{a-2x}{a+2x} \div \frac{a^2-4x^2}{a^2+4x^2}$.

12. $\frac{(x+3)(x+5)}{(x-1)(x-4)} \div \frac{(x-1)(x+3)}{(x-4)(x+5)}$.

13. $\frac{x^2-xy}{y^2} \div \frac{xy-y^2}{x^2}$.

14. $\frac{2x-4}{y^3} \div \frac{x-2}{x^3}$.

15. $\left(1-\frac{x^2}{y^2}\right) \div \left(1-\frac{x}{y}\right)$.

16. $\frac{5x^2-5y^2}{a+b} \div \frac{10x-10y}{a+b}$.

REDUCTION OF COMPLEX FRACTIONS.

110. A complex fraction may be regarded as a case of division of fractions and treated in the same way.

***Rule.**—Invert the denominator and multiply it by the numerator.*

Reduce the following complex fractions to simple ones:

WRITTEN EXAMPLES.

1. $\dfrac{\frac{x}{y}}{\frac{a}{b}}$.

2. $\dfrac{\frac{3}{x}}{\frac{a}{y}}$.

3. $\dfrac{\frac{a+b}{c}}{\frac{a}{a-b}}$.

4. $\dfrac{a \times \frac{x}{y}}{x + \frac{y}{4}}$.

5. $\dfrac{\frac{a}{b} \times \frac{b}{c}}{\frac{x+y}{x} \times \frac{x}{y}}$.

6. $\dfrac{a - \frac{x^2}{a}}{1 - \frac{x}{a}}$.

7. $\dfrac{\frac{mn}{xy}}{\frac{x-y}{m}}$.

8. $\dfrac{4\frac{2}{3}}{1 + \frac{a}{b}}$.

9. $\dfrac{1 + \frac{1}{x}}{x - \frac{1}{x}}$.

10. $\dfrac{\frac{a}{a-b} - 1}{\frac{b^2}{a^2 - b^2}}$.

11. $\dfrac{\frac{a^2 + 2ac + c^2}{a - c}}{\frac{a^2 - c^2}{2c}}$.

12. $\dfrac{\frac{x-y}{x+y} \times \frac{x^2 - y^2}{x^2}}{\frac{x+y}{y^2}}$.

13. $\dfrac{a + b + \frac{b^2}{a-b}}{\frac{ab}{a^2 - b^2}}$.

14. $\dfrac{1 - \frac{b}{a+b}}{1 + \frac{b}{a-b}}$.

15. $\dfrac{\frac{a^2 + 2ax + x^2}{a^2 + x^2}}{\frac{a^2 - x^2}{a^2 + x^2}}$.

SECTION XLVI.

MULTIPLICATION OF MIXED QUANTITIES.

111. It is often necessary to multiply mixed quantities by entire ones, without reducing the former to fractions.

Ex. Multiply $3x + \frac{4}{y} + 6$ by 2.

Solution. Multiplying each term separately, we have

$$6x + \frac{8}{y} + 12.$$

So $\left(5 - \frac{x}{6}\right) \times 3 = 15 - \frac{x}{2}$, and $\left(\frac{6x}{5} + 7\right) \times 10 = 12x + 70.$

Note.—When the denominator is contained exactly in the multiplier, the result will be an entire quantity. To find it, divide the denominator into the multiplier, and multiply the result by the numerator.

Perform the following indicated multiplication:

MENTAL EXERCISE.

1. $\left(\frac{a}{2x} + \frac{4}{x} + 6\right) \times 3.$
2. $\left(\frac{2x}{3} + \frac{5y}{4} + 9\right) \times 4.$
3. $\left(\frac{3x}{5} + \frac{y}{2}\right) \times 10.$
4. $\left(2 + \frac{3x}{7} + \frac{5x}{2}\right) \times 14.$
5. $\left(\frac{7}{2x} + \frac{3}{4x} + 5\right) \times 4x.$
6. $\left(1 + \frac{2}{x} + \frac{3}{x^2}\right) \times x^2.$
7. $\left(\frac{x+3}{4} + 5x\right) \times 8.$
8. $\left(\frac{2x-1}{2} + \frac{4x+3}{3}\right) \times 6.$
9. $\left(\frac{5}{x+3} + 3\right) \times (x+3).$

10. $\left(\frac{1}{x-1}+\frac{1}{x+2}\right)\times(x-1)(x+2)$.

Note.—If a fraction is preceded by the sign −, and the denominator is removed by the multiplication, the numerator must have all its signs changed, because it is to be subtracted. But the signs must not be changed unless the denominator is removed.

Thus,
$$\left(3x-\frac{2x+1}{3}\right)\times 6 = 18x-4x-2;$$
$$\left(7-\frac{5+x}{3}\right)\times 2 = 14-\frac{10+2x}{3}.$$

11. $\left(2x-\frac{3+x}{4}\right)\times 4$.

13. $\left(2y-\frac{3y-x}{3}\right)\times 12$.

12. $\left(4x+\frac{x+2y}{5}\right)\times 10$.

14. $\left(\frac{x+y}{2}-\frac{x-y}{4}\right)\times 4$.

15. $\left(4x-\frac{x+1}{2}+\frac{x-2}{3}\right)\times 2$.

16. $\left(\frac{1}{2x-1}-\frac{1}{2x+1}\right)\times(2x-1)(2x+1)$.

17. $\left(\frac{3}{2x}-\frac{2}{x-3}\right)\times 2x(x-3)$.

18. $\left(\frac{a-b}{a+b}-\frac{a+b}{a-b}\right)\times(a-b)(a+b)$.

112. It will be observed that the denominators all cancel out when the multiplier is a multiple of them all.

SECTION XLVII.

REVIEW QUESTIONS.

MENTAL EXERCISE.

1. WHAT is a multiple of a quantity?

2. Give a multiple of $5ax^3$.

3. What is the least common multiple of two or more quantities?

4. What is the least common multiple of $9ax$, $3a^3y$, and $4ay^2$?

5. Of $a^2(a+x)$, $ax(a-x)^2$, and $x^2(a-x)(a+x)$?

6. Of xy^2, $y(x+y)^2$, and $x^3(x+y)^3$?

7. What is the rule for finding the least common multiple of such factored expressions?

8. An algebraic fraction being regarded as a case of indicated division, what parts correspond to the dividend and divisor?

9. What corresponds to the quotient?

10. In the fraction $\frac{a^2-b^2}{a-b}$ what is the divisor, what the dividend, and what is the quotient?

11. How are fractions reduced to their lowest terms?

12. How is the least common denominator of two or more fractions found?

13. How are the new numerators found?

14. Collect, without reducing, $a+\frac{x}{y}+3a-\frac{4x}{y}$.

15. Collect in the same way $\frac{4}{a}+\frac{3}{x}+\frac{5}{a}-\frac{8}{x}$.

16. Collect $\frac{x}{2} + \frac{x}{3} + \frac{y}{4} - \frac{y}{5}$.

17. Add $\begin{array}{l}\frac{5}{x} + \frac{5}{y} \\ \underline{\frac{7}{x} - \frac{3}{y}}\end{array}$

18. Subtract $\begin{array}{l}\frac{8}{x} + \frac{8}{y} \\ \underline{\frac{8}{x} + \frac{12}{y}}\end{array}$

19. Subtract $\begin{array}{l}\frac{6}{x} + \frac{9}{y} \\ \underline{\frac{6}{x} + \frac{4}{z}}\end{array}$

20. Reduce $a - \frac{b^2 - bc}{a}$ to a fraction, explaining the effect of the sign $-$ before the fraction.

21. Multiply $5x - \frac{2y + 4}{3}$ by 6, explaining the effect of the sign $-$ before the fraction.

22. Why does $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$?

23. Why does $\frac{a}{c} \times b = \frac{ab}{c}$?

24. When does a fraction, multiplied by an entire quantity, give an entire quantity as the product? (Art. 112.)

SECTION XLVIII.

CLEARING EQUATIONS OF FRACTIONS.

113. Given $4 + \frac{x}{2} + \frac{x}{5} = 11$, to find the value of x.

Solution. We found in Art. 112 that when a mixed quantity was multiplied by a quantity that was a multiple of all the denominators, these were cancelled out, and the product contained no fractions. Hence, if we multiply the first member of the equation, $4 + \frac{x}{2} + \frac{x}{5}$, by 10, the least common multiple of the denominators, the result will be an entire quantity. To preserve the equality, the second member, 11, must also be multiplied.

The whole solution would be as follows:

Clearing of fractions by multiplying each term by 10, the equation becomes $40 + 5x + 2x = 110.$

Transposing, $5x + 2x = 110 - 40.$

Collecting, $7x = 70.$

Dividing by 7, we find $x = 10.$

Proof. Substituting this value in the first equation, it becomes $4 + 5 + 2 = 11.$

Collecting, $11 = 11.$

Ex. Given $\frac{x+1}{4} - \frac{2x-5}{3} = \frac{x-15}{8}$, to find the value of x.

Solution. The least common multiple of the denominator is 24.

Clearing of fractions, by multiplying each term by this, and taking care to change signs in the second numerator, the equation becomes $6x + 6 - 16x + 40 = 3x - 45.$

Which being solved, gives $x = 7.$

Proof. Substituting this value in the first equation,

$$\frac{8}{4}-\frac{9}{3}=\frac{-8}{8}.$$

Reducing and collecting, $-1=-1.$

***Rule.**—Find the least common multiple of the denominators, and multiply each term of the equation by it.*

CAUTION.—On clearing an equation of fractions be careful to change all the signs in the numerators of those which are preceded by the sign −.

Solve the following equations:

WRITTEN EXAMPLES.

As a mental exercise let the student clear of fractions the first 16 equations, stating the least common multiple of the denominators, and giving the equation after multiplication.

1. $\frac{3x}{4}+5x-7=4x.$
2. $x+\frac{4-x}{2}=\frac{2+4x}{6}.$
3. $x+\frac{x}{3}-\frac{7x}{9}=\frac{20}{9}.$
4. $5x-\frac{5x-20}{4}=4x+3.$
5. $x-\frac{2x+1}{3}=\frac{x+7}{5}.$
6. $\frac{x+3}{8}-\frac{x-2}{4}=\frac{6-x}{4}.$
7. $\frac{3x-10}{4}-\frac{2x+3}{3}=9-2x.$
8. $\frac{2x-1}{3}-\frac{x+4}{9}=\frac{5x-1}{27}.$
9. $\frac{4}{x}+5=7.$
10. $\frac{4}{x}+\frac{14}{x}=2.$
11. $\frac{3}{x}+\frac{5}{2x}+\frac{5}{8}=\frac{8}{x}.$
12. $\frac{4}{x-8}=1.$
13. $\frac{30}{2x+5}=\frac{8}{x-1}.$
14. $\frac{36}{5x+8}=\frac{4}{x}.$
15. $\frac{2(x-2)}{3x-2}=\frac{5}{8}.$
16. $\frac{3}{x-4}=\frac{6}{x+4}.$

17. $3x + \dfrac{3x - 5}{4} = \dfrac{2x + 6}{5} + 21.$

18. $\dfrac{2x + 3}{6} - \dfrac{5x - 7}{7} = \dfrac{x}{6} - \dfrac{x - 2}{7}.$

19. $\dfrac{3(x + 9)}{4} + 2x - 7 = 14 - \dfrac{7x + 15}{6}.$

20. $\dfrac{3x + 5}{8} - \dfrac{21 + x}{3} = 39 - 5x.$

21. $x - \dfrac{x - 2}{3} = \dfrac{50 + x}{5} - \dfrac{x + 23}{4}.$

22. $\frac{1}{2}(x - 1) - \frac{2}{3}(x + 2) + \frac{1}{4}(x - 3) = 4.$

After clearing of fractions in the following equations the terms which contain x^2 may be cancelled out.

NOTE.—If the denominators have no common multiple less than their product, the operation of the rule for clearing of fractions is equivalent to multiplying the numerator of each fraction by all the denominators except its own, and each entire term by all the denominators. Thus, the 25th would be cleared of fractions by multiplying the first numerator by the second denominator, and the second numerator by the first denominator. This naturally shortens the work when the denominators are large.

23. $\dfrac{x^2}{x - 4} = x + 6.$

24. $\dfrac{x^2}{x - 4} = x - 4.$

25. $\dfrac{2x + 3}{3x + 9} = \dfrac{2x - 8}{3x - 13}.$

26. $\dfrac{6x + 7}{3x + 1} = \dfrac{2x + 19}{x + 7}.$

27. $\dfrac{5x + 3}{x - 1} + \dfrac{2x - 3}{2x - 1} = 6.$

28. $\dfrac{4}{3 + x} - \dfrac{1}{x} = \dfrac{9}{7x}.$

SECTION XLIX.

EXERCISES IN THE FORMATION AND SOLUTION OF FRACTIONAL EQUATIONS.

WRITTEN EXAMPLES.

1. If $\frac{1}{3}$ of $x+1$ minus $\frac{1}{5}$ of $3x-1$ equals $x-2$, what is the value of x?

2. If $\frac{1}{2}$ of $19-2x$ subtracted from $2x$ equals $\frac{1}{3}$ of $2x-11$, what is the value of x?

3. If 2 added to $\frac{1}{5}$ of $3x-2$ equals $\frac{2}{3}$ of $x+2$, what is the value of x?

4. If $\frac{1}{2}$ of $x-1$ is 1 more than $\frac{1}{3}$ of x, what is the value of x?

5. The fifth part of a number, plus the fourth part, is 1 less than half the number. What is the number?

6. $\frac{3}{7}$ of a certain number, minus $\frac{1}{4}$ of it, is equal to 5. What is the number?

7. $\frac{3}{5}$ of one number is equal to $\frac{1}{2}$ of a number 3 greater than the first. What is the first number?

8. There are 3 consecutive numbers, such that $\frac{1}{6}$ of the first, $\frac{2}{5}$ of the second, and $\frac{3}{8}$ of the third are together 2 less than the first number. What are the numbers?

9. There are 3 consecutive even numbers, such that $\frac{2}{3}$ of the first, plus $\frac{3}{4}$ of the second, minus $\frac{4}{5}$ of the third, is equal to 2. What are the numbers?

10. $\frac{1}{2}$ of one number is 1 greater than $\frac{1}{7}$ of a number 5 greater than twice the first number. What is the number?

11. $\frac{1}{3}$ of one number, minus $\frac{1}{5}$ of a number 4 greater than the first, is equal to 2. What is the first number?

12. $\frac{1}{7}$ of 5 more than a certain number is equal to the number minus 7. What is the number?

SECTION L.

STATEMENT AND SOLUTION OF PROBLEMS PRODUCING FRACTIONAL EQUATIONS.

MENTAL EXERCISE IN THE STATEMENT OF PROBLEMS.

1. IF a man has a certain number of sheep, and sells $\frac{1}{4}$ of them, what fractional part will be left?

2. If he has x sheep, and sells $\frac{1}{4}$ of them, what expression represents the number left?

3. If he has $\frac{3}{4}(x - 10)$, and sells $\frac{1}{3}$ of them, what expression represents the number left?

NOTE.—If he sells $\frac{1}{3}$ of them, he will have $\frac{2}{3}$ left. So we find $\frac{2}{3}$ of $\frac{3}{4}(x - 10)$. This is much easier than to find $\frac{1}{3}$ of $\frac{3}{4}(x - 10)$, and then to subtract that result.

4. If 6 people hire a boat for x dollars, what represents the share of each?

5. If 8 people hire a boat for the same sum, what represents the share of each?

6. If the difference between the shares is $1, what equation can be made?

7. If a man walks x miles at 4 miles an hour, what expression represents the number of hours occupied?

8. If a man walks x miles at 3 miles an hour, stops 2 hours, and then walks 6 miles, what will represent the whole time occupied?

9. If during the same time another man walks $x - 6$ miles at 2 miles an hour, what equation can be made?

10. If a man agrees to work for a year, for $x + 100$ dollars, what represents one month's wages?

11. If he receives x dollars for 7 months' work, what represents one month's wages?

12. What equation can be made, if the wages in both cases are the same?

SOLUTION OF PROBLEMS PRODUCING FRACTIONAL EQUATIONS.

114. Most of the difficult points in the solution of the following problems have been presented in a simple form in the preceding mental exercise. The student should refer to these if he need help. He should also refer to the rule and remarks found on pages 43 and 44. Let especial attention be paid to *neatness of explanation.*

PROB. A man spends \$50 of his money for a harness, $\frac{1}{5}$ of the remainder for a horse, and $\frac{1}{4}$ of what still remains for a carriage. He then has left \$35 more than half his money. How much had he at first?

STATEMENT. Let x = the number of dollars the man had;

then $x - 50$ = the number in first remainder, *because he spent* \$50 *for the harness;*

and $\frac{4(x-50)}{5}$ = the number in second remainder, *because he spent* $\frac{1}{5}$ *of the first remainder for the horse;*

and $\frac{3(x-50)}{5}$ = the number in final remainder, *because he spent* $\frac{1}{4}$ *of the former remainder for the carriage.*

Also, $\frac{x}{2} + 35$ = the final remainder, *because there was left* \$35 *more than half the first amount.*

And because these two remainders represent the same quantity, we have the equation $\frac{3(x-50)}{5} = \frac{x}{2} + 35.$

SOLUTION. *Clearing of fractions by multiplying all the terms by* 10, $6x - 300 = 5x + 350.$

Transposing and collecting, $x = 650.$

Proof. $650 - 50 = 600$; $\frac{4}{5}$ of $600 = 480$, $\frac{3}{4}$ of $480 = 360$, $360 = \frac{650}{2} + 35.$

PROB. Three men contributed money for a business enterprise. One gave \$50 less than half the amount, another gave \$25 more than one-fifth of the amount, and the third gave \$50 more than one-fourth of it. What was the whole amount?

STATEMENT. Let x = the number of dollars in the whole amount;

then $\frac{x}{2} - 50$ = the number of dollars first gave, *because he gave \$50 less than one-half the amount;*

and $\frac{x}{5} + 25$ = the number of dollars the second gave, *because he gave \$25 more than* $\frac{1}{5}$ *of the amount;*

and $\frac{x}{4} + 50$ = the number of dollars the third gave, *because he gave \$50 more than* $\frac{1}{4}$ *of the amount.*

Since the sum of these three amounts will give the whole amount contributed, which amount is also represented by x, this sum must be equal to x. Therefore, we have the equation

$$\frac{x}{2} - 50 + \frac{x}{5} + 25 + \frac{x}{4} + 50 = x.$$

The solution may now be easily made.

PROBLEMS PRODUCING FRACTIONAL EQUATIONS.

1. John and James together have 30 cents. James has 2 more than $\frac{3}{4}$ as many as John. How many has each?

2. A boy catches, in an afternoon, 1 less than $\frac{3}{5}$ as many fish as he caught in the morning. Altogether, he caught 23 fish. How many did he catch in the morning?

3. A boy visited his traps one morning and found $\frac{3}{4}$ of them sprung, but 2 of these empty. The number of animals caught was 1 more than half the number of traps. How many traps had he?

4. A is 8 years older than B. $\frac{1}{7}$ of A's age is equal to $\frac{1}{5}$ of B's. What are the ages of each?

5. A son's age is $\frac{2}{3}$ of that of his father; 20 years ago his age was half that of the father. What are their ages now?

6. One year before the Declaration of American Independence, 1776, Thomas Jefferson's age was $\frac{4}{5}$ that of John Adams. They died upon the same day, just 50 years after the Declaration, when their united ages amounted to 174 years. How old were they at the Declaration of Independence, and at their death?

7. Three boys contributed different amounts to buy a ball. One gave 2 cents less than $\frac{1}{2}$ the whole amount; another gave 5 cents more than $\frac{1}{4}$ of it, and the third 1 cent less than $\frac{1}{5}$ of it. What was the cost of the ball?

8. A farmer had sheep in 3 fields. In the second there were twice as many as in the first; and in the third, 6 more than in the second. He sold 44 sheep, taking $\frac{1}{2}$ of those in the first field, $\frac{3}{4}$ of those in the second, and $\frac{2}{3}$ of those in the third. How many sheep were in each field at first?

9. A had 4 times as many dollars as B; but when A paid B $500, he had $50 less than $\frac{1}{2}$ B's amount. How much had each at first?

10. A and B had equal sums of money; but B spent $50, and then received from A $150, when A had $\frac{3}{8}$ as much money as B. How much had each at first?

11. Four persons hired a carriage for a certain sum. Two persons more having joined the party, each had to pay \$$\frac{1}{2}$ less than they had expected. For how many dollars had they hired the carriage?

12. A party of 10 persons charter a boat for a certain sum, but 3 having withdrawn, each of the remainder has his share

of the expense increased \$6. For what sum had they chartered the boat?

13. A man walks from Philadelphia to New York at the rate of 24 miles a day. He returns at the rate of 20 miles a day, and is $\frac{4}{5}$ of a day longer in returning than in going. What is the distance between the cities?

14. A man rides 14 miles, and walks back at half the rate per hour that he rode, being gone 6 hours. How many miles per hour did he walk and ride?

SUGGESTION.—Let x = number of miles an hour that he walked.

15. One man starts on foot and another on horseback for a town 12 miles distant. The horseman travels twice as fast as the other. He arrives at the town, spends half an hour there, and meets the other after returning 3 miles. At what rates did they travel?

SUGGESTION.—Find an equality between the time spent by each on the journey.

16. A sailing-vessel starts from New York for Glasgow, distant 3000 miles, at the same time as a steamer. The steamer goes 3 times as fast as the sailing-vessel. It reaches its port, spends 12 days in discharging its cargo and reloading, and meets the sailing-vessel on the return voyage 600 miles from Glasgow. How many miles per day did each vessel go?

17. A man spends \$400 of his salary for board, $\frac{1}{4}$ of the remainder for clothes, and $\frac{1}{3}$ of what still remains for general expenses. The balance, \$400, he saves. What is his salary?

18. A person had a certain sum with which to buy a horse, carriage, and harness. He spent \$25 for the harness, \$25 less than $\frac{2}{3}$ of the remaining money for the carriage, and $\frac{2}{3}$ of the same remainder for the horse. How much did he spend for all?

19. A man left by will $3000 to his daughter, $1000 more than $\frac{1}{2}$ the remainder of his money to his elder son, and $3000 to the younger son. How much did he leave in all?

20. A mechanic agreed to work for a year for $285 and a chest of tools. At the end of 7 months he stopped work and received the chest of tools and $135. At what was the chest of tools valued?

21. A man walks 14 miles in the morning, and then quickens his pace 1 mile an hour, and walks 18 miles in the same time in the afternoon. At what rates does he walk?

22. What is the fraction whose denominator is 2 more than 3 times the numerator, and which is equal to $\frac{2}{7}$?

23. The denominator of a certain fraction is twice its numerator, and if 1 be subtracted from the numerator and 2 added to the denominator, the fraction will be equal to $\frac{1}{4}$.

24. At a certain election there were 3 candidates. The first received 10 more than $\frac{1}{2}$ the number of votes cast; the second received $\frac{2}{3}$ as many as the first; and the third, 189 votes. How many votes were cast?

25. A man buys a horse from a farmer, agreeing to pay him $20, and work for him 4 months. He continues to work for 6 months, and then receives the horse and $20 besides. What is the value of the horse?

26. A and B start at the same time for a day's journey, having only one saddle-horse. A can walk 4 miles per hour, B can walk 3, and the horse can travel 6. A first rides the horse for a while, then ties him by the roadside and walks on. B walks till he comes to the horse, mounts him, and travels after A, overtaking him 8 hours after their first start. How many hours did each person ride the horse?

SECTION LI.

SIMPLE EQUATIONS.

DEFINITIONS.

115. AN algebraic term, it will be remembered, is an algebraic expression having no separate parts connected by the signs + and −.

116. A term is said to have different **degrees**, according to the **number of letters** it contains.

A term containing **one** letter is of the **First Degree**; as, $4x$.

A term containing **two** letters is of the **Second Degree**; as, $6ab$ or $8y^2$.

A term containing **six** letters is of the **Sixth Degree**; as, a^3xy^2, $4a^4x^2$, or $3c^6$.

The degree of a term can always be found by adding together the exponents of all of its letters.

Terms that have the same degree are said to be **homogeneous terms.** Thus, a^2, $2ab$, and b^2 are homogeneous terms, each containing two letters. So also are x^4, $2x^2y^2$, and y^4, each being of the fourth degree.

117. If each term of an *equation* contains not more than one unknown letter, the equation is said to be of the **First Degree.**

Equations of the First Degree are also called **Simple Equations.**

$4x + 1 = 7$, $x + 3y = 8$, and $4x + 5y + 2z = 16$ are simple equations, while $x^2 = 16$, $x + xy = 12$ are not simple equations.

Equations are also distinguished by the number of unknown quantities which they contain. Thus, $4x + 2 = 18$ is a simple equation of *one unknown quantity;* $2x - y = 12$ is a simple equation of *two unknown quantities;* and so on.

SECTION LII.

SOLUTION OF SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.—ELIMINATION BY SUBSTITUTION.

118. GIVEN $5x - 2y = 19$,

$3x + 2y = 21$, to find the values of x and y.

If we take these equations separately we cannot find anything about the value of x or y. If either equation contained only x or y, we could readily solve it. We found in Section XX. that two equations like the above could always be combined by adding or subtracting, so as to cancel the terms containing one of the two letters. Thus, if we add the members of these equations, we find $8x = 40$. This is an equation containing only one letter, x. The other letter, y, has disappeared during the process of addition. This operation of cancelling out, or getting rid of, one letter was called **Elimination.**

Elimination may be defined as the process of combining two equations so as to obtain a resulting equation in which one of the letters does not occur.

There are three ordinary methods of **elimination**—by **addition or subtraction,** by **substitution** and by **comparison.**

Elimination by addition or subtraction was explained in Section XX.

Solve the following pairs of equations by this method of elimination:

1. $\begin{Bmatrix} 3x + 4y = 29 \\ 2x - 5y = 4 \end{Bmatrix}$.   2. $\begin{Bmatrix} 4x - y = 27 \\ 3x + 2y = 34 \end{Bmatrix}$.

ELIMINATION BY SUBSTITUTION.

119. Given $x + 4y = 24$,
and $x = 4$, to find the value of y.

SOLUTION. Substituting 4 for x in the first equation, we have

$$4 + 4y = 24.$$

This equation contains only y; the x has been eliminated by the **substitution** of its value.

The value of y may now be readily found.

Ex. Given $2x + y = 15$,
$y = 4x - 21$, to find the values of x and y.

SOLUTION. Here, as before, we substitute the value of x, in this case $(4x - 21)$ enclosing it in a parenthesis, and obtain

$$2x + (4x - 21) = 15.$$

This equation has only one letter, x, in it. The letter y was **eliminated** by the **substitution** of its value. The parenthesis can be removed, and the value of x found, and then the value of y, by substitution.

Ex. Given $3x - y = 5$,
$y = 4x - 7$, to find the values of x and y.

SOLUTION. Here, as before, we enclose the value of y in a parenthesis, and substitute for y in the first equation, obtaining

$$3x - (4x - 7) = 5.$$

The parenthesis can be removed, care being taken to note the effect of the sign $-$ before it, and the value of x found. The value of y may be found as before.

Solve the following equations by this method of substitution:

WRITTEN EXAMPLES.

1. $\left\{\begin{aligned} 4x + y &= 12 \\ y &= 5x - 6 \end{aligned}\right\}.$ 2. $\left\{\begin{aligned} x + 3y &= 26 \\ x &= 19 - 2y \end{aligned}\right\}.$

NOTE.—In the following equations clear of fractions *after* substituting:

3. $\left\{\begin{aligned} 4x + y &= 40 \\ y &= \frac{21 - x}{3} \end{aligned}\right\}.$ 4. $\left\{\begin{aligned} 7x - y &= 32 \\ y &= \frac{9x - 4}{5} \end{aligned}\right\}.$

Ex. Given $5x - 7y = 14$,

$$y = \frac{3x + 3}{8}, \text{ to find the values of } x \text{ and } y.$$

SOLUTION. The value of y in this case is the fraction $\frac{3x + 3}{8}$. Enclosing it in a parenthesis, and substituting it for y in the first equation, we have

$$5x - 7\left(\frac{3x + 3}{8}\right) = 14.$$

This equation may be cleared of fractions and the value of x found, and then the value of y, as before.

In the same way solve the following pairs of equations:

WRITTEN EXAMPLES.

5. $\left\{\begin{aligned} 10x - 4y &= 14 \\ y &= \frac{11x - 1}{6} \end{aligned}\right\}.$ 6. $\left\{\begin{aligned} 7x + 2y &= 37 \\ x &= \frac{4y - 11}{7} \end{aligned}\right\}.$

120. Given $\left\{\begin{aligned} 3x - 4y &= 24 \\ 5x + 3y &= 11 \end{aligned}\right\}$, to find x and y.

SOLUTION. Here we have no value for y. But by transposing $5x$ in the second equation, obtaining

$$3y = 11 - 5x,$$

and dividing by 3, we have $y = \frac{11 - 5x}{3}$.

Substituting the fraction $\frac{11 - 5x}{3}$ for y in the first equation, it becomes

$$3x - 4\left(\frac{11 - 5x}{3}\right) = 24.$$

The values of x and y can now be found as before.

For elimination by substitution we have the following

Rule.—*Transpose the term containing one of the unknown quantities in one of the equations to the second member, and divide by the coefficient of the other unknown quantity. Substitute the expression thus found for that quantity in the other equation.*

Find the values of x and y in the following pairs of equations, eliminating by the method of substitution.

WRITTEN EXAMPLES.

1. $\left\{\begin{array}{l} 5x - 2y = 22 \\ 4x + 3y = 13 \end{array}\right\}.$

2. $\left\{\begin{array}{l} 7x + 5y = 64 \\ 4x - 2y = 22 \end{array}\right\}.$

3. $\left\{\begin{array}{l} 8x - 2y = 23 \\ 5x + 6y = 18 \end{array}\right\}.$

4. $\left\{\begin{array}{l} x + 3y = 24 \\ 4x + 2y = 56 \end{array}\right\}.$

5. $\left\{\begin{array}{l} 4x + 7y = 19 \\ 2x - 4y = -28 \end{array}\right\}.$

6. $\left\{\begin{array}{l} 2y - 4x = -32 \\ 3x + 5y = 50 \end{array}\right\}.$

SECTION LIII.

ELIMINATION BY COMPARISON.

121. Given $\left\{\begin{array}{l} x = 4y - 3 \\ x = 6y - 7 \end{array}\right\}$, to find the values of x and y.

Solution. From the first equation we find that the expression $4y - 3$ represents the value of x; from the second we find that $6y - 7$ also represents the value of x. These expressions are therefore equal to each other; that is, $4y - 3 = 6y - 7$. As this is an equation containing only one unknown letter, it can be readily solved. The letter x has been eliminated by the comparison of the two values of x.

Transposing the terms, the equation becomes

$$4y - 6y = 3 - 7.$$

Collecting, $-2y = -4.$

Dividing by -2, we find $y = 2.$

Substituting this value in the first equation, $x = 8 - 3.$

Collecting, we find $x = 5.$

Proof. Substituting the values of x and y in the original equations,

$$5 = 8 - 3.$$

$$5 = 12 - 7.$$

Solve the following pairs of equations, eliminating by comparison:

WRITTEN EXAMPLES.

1. $\begin{Bmatrix} x = 4y + 24 \\ x = 2y + 14 \end{Bmatrix}.$

2. $\begin{Bmatrix} y = 23 - 3x \\ y = x - 5 \end{Bmatrix}.$

3. $\begin{Bmatrix} x = \dfrac{9y + 3}{2} \\ x = 4y + 2 \end{Bmatrix}.$

4. $\begin{Bmatrix} x = \dfrac{5y + 16}{4} \\ x = \dfrac{12y - 3}{5} \end{Bmatrix}.$

122. Given $\begin{Bmatrix} 6x - 8y = 28 \\ 2x + 3y = 32 \end{Bmatrix}$, to find the values of x and y.

Solution. If we transpose the terms containing one letter, as y, the equations become

$$6x = 28 + 8y,$$

$$2x = 32 - 3y.$$

Dividing the members of each by the coefficient of x in that equation, we have

$$x = \frac{28 + 8y}{6}.$$

$$x = \frac{32 - 3y}{2}.$$

These values of x can now be compared, and the numerical values of x and y found as in the preceding examples.

Rule.*—Transpose the terms of both equations, so as to leave in the first members one unknown quantity, the same in each equation. Divide by the coefficients of this quantity. Place the values of the unknown quantity thus found equal to each other, and solve the resulting equation.*

Solve the following pairs of equations, eliminating one letter by **comparison**:

WRITTEN EXAMPLES.

1. $\left\{\begin{array}{r} 2x - y = 22 \\ x + 2y = 21 \end{array}\right\}.$

2. $\left\{\begin{array}{r} 4x + 9y = 31 \\ 2x - 6y = 12 \end{array}\right\}.$

3. $\left\{\begin{array}{r} 7x - 4y = 11 \\ 3x + 2y = 27 \end{array}\right\}.$

4. $\left\{\begin{array}{r} 3x + 2y = 16 \\ x - 4y = 24 \end{array}\right\}.$

5. $\left\{\begin{array}{r} 5x - 2y = -2 \\ 8x + 3y = 34 \end{array}\right\}.$

6. $\left\{\begin{array}{r} 8x + 2y = 10 \\ 4x - 7y = -11 \end{array}\right\}.$

SECTION LIV.

GENERAL EXERCISE IN SOLVING EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

Solve each of the following pairs of equations by all the three methods:

Note.—If the same results are obtained each time, it will be sufficient proof of the correctness of the work.

WRITTEN EXAMPLES.

1. $\left\{\begin{array}{r} 4x - 7y = 34 \\ 2x - 9y = 6 \end{array}\right\}.$

2. $\left\{\begin{array}{r} 3x + 2y = 5 \\ x + 4y = -15 \end{array}\right\}.$

3. $\left\{\begin{array}{r} 7x + 16y = 59 \\ 4x - 8y = 23 \end{array}\right\}.$

4. $\left\{\begin{array}{r} 2x - 5y = 0 \\ 3x - 4y = 28 \end{array}\right\}.$

123. If equations containing two unknown quantities are not in the right form for elimination, they must be reduced to it.

Ex. Given $\left\{\begin{array}{l} \dfrac{3x+5}{4}+y=5+\dfrac{3+2y}{3} \\ 2x-4-(4x-5y)=1 \end{array}\right\}$, to find x and y.

SOLUTION. Clearing the first equation of fractions by multiplying each term by 12, we have

$$9x+15+12y=60+12+8y.$$

Transposing, $\quad 9x+12y-8y=60+12-15.$

Collecting, $\quad 9x+4y=57. \qquad (3)$

Performing the indicated subtraction in the second equation, we have $\quad 2x-4-4x+5y=1.$

Transposing and collecting, $\quad -2x+5y=5. \qquad (4)$

Equations marked (3) and (4) are now in the regular form for elimination, and either method may be followed, although it is generally best to use that of addition or subtraction.

Solve the following pairs of equations:

WRITTEN EXAMPLES.

1. $\left\{\begin{array}{r} \dfrac{3x-y}{2}+2x=27 \\ 4x-5y=22 \end{array}\right\}.$

2. $\left\{\begin{array}{r} \dfrac{20-2x}{y}=\quad -2 \\ x+2y=x-4 \end{array}\right\}.$

3. $\left\{\begin{array}{l} \dfrac{3}{2x+1}=\dfrac{2}{3y-9} \\ 2y-x=\dfrac{4y+x}{4} \end{array}\right\}.$

4. $\left\{\begin{array}{l} \dfrac{x+2y}{2x+y}=\dfrac{1}{14} \\ 3x+5y=7 \end{array}\right\}.$

5. $\left\{\begin{array}{r} \dfrac{3x-2y}{5}+\dfrac{x-y}{2}=1 \\ \dfrac{x}{3}+\dfrac{y}{2}=4 \end{array}\right\}.$

6. $\left\{\begin{array}{l} \dfrac{2x+3y}{16}+\dfrac{x}{12}=2\frac{3}{4} \\ \dfrac{7y-3x}{2}-2y=3 \end{array}\right\}.$

7. $\left\{\begin{array}{l}\dfrac{3x-6y}{x+4y}=\dfrac{3}{5}\\ \dfrac{8}{2x+4}=\dfrac{4}{x+y}\end{array}\right\}$. 9. $\left\{\begin{array}{l}\dfrac{x}{3-y}=\dfrac{1}{2}\\ \dfrac{4}{y-x}=-\dfrac{1}{3}\end{array}\right\}$.

8. $\left\{\begin{array}{l}\dfrac{3x+9}{2}=\dfrac{x+y}{4}\\ \dfrac{5x+y}{y-2x}=-\dfrac{1}{6}\end{array}\right\}$. 10. $\left\{\begin{array}{l}\dfrac{4x-10y}{10y-x}=11\\ \dfrac{3x-12}{6y}=2\end{array}\right\}$.

11. $\left\{\begin{array}{l}\dfrac{5y+3}{6}+x=3+\dfrac{4+x}{2}\\ \dfrac{2x+y}{11}=\dfrac{3y-7}{2}\end{array}\right\}$.

12. $\left\{\begin{array}{l}8x-\dfrac{2x+5}{9}=11y-3\\ \dfrac{4x-2y}{7}=3+\dfrac{5x-6y-2}{5}\end{array}\right\}$.

13. $\left\{\begin{array}{l}\dfrac{4x+5y}{40}=x-y\\ \dfrac{2x-y}{3}+2y=\dfrac{1}{2}\end{array}\right\}$.

124. Given $\left\{\begin{array}{l}\dfrac{6}{x}+\dfrac{8}{y}=5\\ \dfrac{2}{x}+\dfrac{4}{y}=2\end{array}\right\}$, to find the values of x and y.

Solution. Equations of this kind are most easily solved by eliminating before clearing of fractions.

Writing the first equation, $\dfrac{6}{x}+\dfrac{8}{y}=5.$

Multiplying the terms of the second equation by 2, $\dfrac{4}{x}+\dfrac{8}{y}=4.$

Subtracting the members, $\frac{2}{x} = 1.$

Clearing of fractions, $2 = x.$

Transposing and changing signs, $x = 2.$

Substituting this value of x in the first equation, $\frac{6}{2} + \frac{8}{y} = 5.$

Reducing, transposing, and collecting, $\frac{8}{y} = 2.$

Clearing of fractions, $8 = 2y.$

Transposing and dividing by -2, $y = 4.$

Proof. Substituting these values in the first two equations, we find

$$\frac{6}{2} + \frac{8}{4} = 5.$$

$$\frac{2}{2} + \frac{4}{4} = 2.$$

Solve the following pairs of equations:

WRITTEN EXAMPLES.

1. $$\left\{\begin{array}{l} \frac{6}{x} + \frac{5}{y} = 1 \\ \frac{9}{x} - \frac{5}{y} = 2 \end{array}\right\}.$$

4. $$\left\{\begin{array}{l} \frac{4}{x} + \frac{9}{y} = 44 \\ \frac{3}{x} + \frac{1}{y} = 10 \end{array}\right\}.$$

2. $$\left\{\begin{array}{l} \frac{4}{x} + \frac{1}{y} = 1 \\ \frac{12}{x} - \frac{1}{y} = 1 \end{array}\right\}.$$

5. $$\left\{\begin{array}{l} \frac{6}{x} + \frac{9}{y} = 3 \\ \frac{10}{x} - \frac{3}{y} = 2 \end{array}\right\}.$$

3. $$\left\{\begin{array}{l} \frac{5}{x} + \frac{3}{y} = 2 \\ \frac{8}{x} + \frac{6}{y} = 3 \end{array}\right\}.$$

6. $$\left\{\begin{array}{l} \frac{4}{x} + \frac{5}{y} = 16 \\ \frac{16}{x} - \frac{1}{y} = 1 \end{array}\right\}.$$

SECTION LV.

EQUATIONS CONTAINING THREE UNKNOWN QUANTITIES.

125. If we know that $2x + y - 3z = 7$, we evidently can tell nothing about the values of the letters.

If we further know that $x - 2y + z = 1$, we can combine the two equations so as to eliminate x by subtraction, and have a resulting equation containing y and z, but cannot tell anything about the value of the quantities.

But if also we know that $3x - 3y + 2z = 8$, we can combine one of the other equations with it, so as to eliminate x, and thus give us a second equation containing only y and z. These two can be combined, and the values of the two letters found, and these values, substituted in either of the original equations, will determine x.

We see from this that when there are three unknown quantities, three equations must be given, and that their solution is very similar to the previous work with two unknown quantities.

The actual operation is as follows:

$$2x + y - 3z = 7; \qquad (1)$$

$$x - 2y + z = 1; \qquad (2)$$

$$3x - 3y + 2z = 8. \qquad (3)$$

Solution. We will first eliminate x, combining equations (1) and (2), and then (2) and (3).

Writing down (1), $\qquad 2x + y - 3z = 7$

Multiplying the members of (2) by 2, $\qquad \underline{2x - 4y + 2z = 2}$

and subtracting, we find $\qquad *5y - 5z = 5 \qquad (4)$

Writing down (3), $3x - 3y + 2z = 8$

Multiplying the members of (2) by 3, $\underline{3x - 6y + 3z = 3}$

and subtracting, we find $*3y - z = 5$ (5)

Taking (4) and (5) as a new set of equations, we see that they can be easily solved: $5y - 5z = 5$ (4)

$3y - z = 5$ (5)

Multiplying the members of (5) by 5, $15y - 5z = 25$

Writing down (4), $\underline{5y - 5z = 5}$

Subtracting, $10y = 20$ (6)

and dividing by 10, we find $*y = 2.$

Substituting this value in (5), $6 - z = 5,$

and changing signs, transposing, and collecting, we find $*z = 1.$

Substituting these two values in (2), $x - 4 + 1 = 1,$

and transposing and collecting, we find $*x = 4.$

Proof. Substituting these values in (1), (2), and (3), we have

$$*8 + 2 - 3 = 7.$$

$$*4 - 4 + 1 = 1.$$

$$*12 - 6 + 2 = 8.$$

126. The student should remember that explaining an example does not consist in merely reading off the work done on the slate or blackboard. The reading of the work is a secondary part, and some of it can often be omitted to advantage. In explaining the preceding solution, if the student will repeat all the explanatory remarks, but only read the equations that are marked with a *, it will be found to give a full and connected explanation, and will only occupy about half the time that would be used in reading the whole. The eye can readily run over the intermediate work.

***Rule.**—I. Combine one of the equations with each of the others, so as to eliminate the same quantity at each combination, and obtain two equations containing not more than two unknown quantities.*

II. Combine these two equations so as to eliminate one of the quantities, and obtain one equation containing only one unknown quantity.

III. Find the value of this by preceding methods. Substitute this value in one of the two equations last used, and find the value of another quantity. Then substitute both these values in one of the original equations, and find the value of the remaining quantity.

NOTE.—Choose for elimination that quantity whose coefficients can be made equal with the least work. Always use the method of Addition or Subtraction, unless one of the others is very evidently the easier.

Solve the following sets of equations:

WRITTEN EXAMPLES.

1. $\left\{\begin{array}{r} x + 2y - z = 9 \\ x - y + 3z = 4 \\ 3x - 2y + z = 7 \end{array}\right\}.$

2. $\left\{\begin{array}{r} 2x - y + 4z = 15 \\ 5x + 2y - z = 16 \\ x + 3y + 2z = 25 \end{array}\right\}.$

3. $\left\{\begin{array}{r} x + y - z = 9 \\ x - y - 2z = -8 \\ 3x - y + 3z = 33 \end{array}\right\}.$

4. $\left\{\begin{array}{r} 5x - 3y + 2z = 9 \\ 2x + 4y - 6z = 16 \\ 3x - 5y + 3z = -7 \end{array}\right\}.$

NOTE.—In the following equations eliminate before clearing of fractions:

5. $\left\{\begin{array}{l} \frac{1}{x} + \frac{4}{y} + \frac{3}{z} = \frac{8}{3} \\ \frac{3}{x} + \frac{2}{y} + \frac{2}{z} = 2 \\ \frac{2}{x} - \frac{3}{y} + \frac{1}{z} = \frac{1}{12} \end{array}\right\}$

127. Although, if there are three unknown quantities, there must be as many equations, yet each equation need not contain all the letters.

Ex. 1. Given $\left\{\begin{array}{rcrl} 4x - 3y + 2z &=& 12 & (1) \\ 3x - 2y &=& 7 & (2) \\ 4y - 3z &=& 10 & (3) \end{array}\right\}$, to find x, y, and z.

Explanation. Here we wish to eliminate so as to obtain two equations containing only two, and those the *same* two, unknown quantities. We observe that eq. (3) has no term containing x. If we combine (1) and (2) so as to eliminate x, we will have another equation containing only y and z. These two can then be combined so as to eliminate one letter and find the value of the other. By substitution, first in one of the last-used equations, and then in one of the original ones, the values of the others can be found.

Let the student solve the above equations in this way:

Ex. 2. Given $\left\{\begin{array}{rcrl} 3x - 2y &=& 16 & (1) \\ 4y - 2z &=& 22 & (2) \\ 2x - 4z &=& 8 & (3) \end{array}\right\}$, to find x, y, and z.

Explanation. Each of these equations contains two unknown quantities, but no two contain the same. We observe that eq. (3) has no term containing y. If we combine (1) and (2) so as to eliminate y, we will have an equation containing only x and z, which with (3) will give two equations containing the same two unknown quantities. Let the student solve the equations in this way.

Solve the following sets of equations:

WRITTEN EXAMPLES.

1. $\left\{\begin{array}{rcr} 3x - 2y + z &=& 0 \\ 2x - 3z &=& 2 \\ 4x + 2y - 6z &=& 18 \end{array}\right\}$.

2. $\left\{\begin{array}{rcr} x + y &=& 19 \\ x + z &=& 18 \\ y + z &=& 17 \end{array}\right\}$.

3. $$\left\{\begin{array}{l} \frac{2}{x}+\frac{3}{y}=\frac{3}{4} \\ \frac{8}{x}+\frac{6}{z}=3 \\ \frac{2}{y}+\frac{2}{z}=1 \end{array}\right\}.$$

4. $$\left\{\begin{array}{r} 4x-3y-2z=12 \\ x+2y \qquad\quad =32 \\ 5z=15 \end{array}\right\}.$$

5. $$\left\{\begin{array}{r} 6x-3y+z=44 \\ 3x-y \qquad =23 \\ x+z=11 \end{array}\right\}.$$

Note.—If the equations are not in the regular form, they must be reduced to it by performing any necessary operations—clearing of fractions, or transposing.

6. $$\left\{\begin{array}{rcl} 3(x+2y) &=& 30+8z \\ 2(2y-3z) &=& 14-x \\ 4(x-z) &=& y-z \end{array}\right\}.$$

8. $$\left\{\begin{array}{rcl} 3(x-y) &=& 6z+9 \\ 2(x-z) &=& 5y+2 \\ 4(y-z) &=& x-3 \end{array}\right\}.$$

7. $$\left\{\begin{array}{l} \frac{2x+y}{3}=2z \\ \frac{3y+z}{3}=3 \\ \frac{x+2z}{7}=y \end{array}\right\}.$$

9. $$\left\{\begin{array}{l} \frac{x+y}{11}=1 \\ \frac{y+z}{4}=2 \\ \frac{x+z}{7}=1 \end{array}\right\}.$$

10. $$\left\{\begin{array}{r} \frac{2x+y}{2}=2z+1 \\ \frac{x+y}{2}-\frac{x-z}{3}=z \\ \frac{2z-y}{4}=x-5 \end{array}\right\}.$$

11. $$\left\{\begin{array}{r} \frac{x+5}{3}-\frac{y+4}{2}=z \\ 2x-y=z+11 \\ 2x=4y-3z+9 \end{array}\right\}.$$

SECTION LVI.

SIMPLE EQUATIONS CONTAINING ANY NUMBER OF UNKNOWN QUANTITIES.

128. We can now give a general rule which will cover the solution of all simple equations containing any number of unknown quantities, provided the equations number as many as the unknown quantities which they contain.

Rule.—I. Combine the equations two and two, taking care to use each equation at least once, so as to eliminate the same unknown quantity, and to produce another set of equations, one less in number than the original set, but containing one less of the unknown quantities.

Note.—If any equation does not contain the unknown quantity selected for elimination, that equation may be immediately included in the second set, and there will be one less elimination necessary.

II. Repeat the operation with the second set, eliminating another unknown quantity and producing a third set of equations, one less in number and containing one less of the unknown quantities. Continue this till only one equation is left containing only one unknown quantity.

III. Find the value of this, and then, by substituting, find the values of the other quantities.

The rule is fully illustrated on the two succeeding pages in the solution of the following example:

Ex. Given $$\left\{\begin{array}{rcr} x-2y+z+u &=& 5\\ 3y-2z+2u &=& 2\\ 2x-4y+3u &=& 5\\ x-y-2z+u &=& -2 \end{array}\right\}$$ to find the values of x, y, z, and u.

FIRST SET, 4 EQUATIONS,	SECOND SET, 3 EQUATIONS,	THIRD SET, 2 EQUATIONS,	1 EQUATION,
Containing 4 unknown quantities, from which we will eliminate x.	Containing 3 unknown quantities, from which we will eliminate y.	Containing 2 unknown quantities, from which we will eliminate u.	Containing 1 unknown quantity.
$x-2y+z+u=5$ (1)	$2z-u=5$ (5)	$2z-u=5$ (5)	$11z=33$ (8)
$3y-2z+2u=2$ (2)	$3y-2z+2u=2$ (2)	$7z+2u=23$ (7)	
$2x-4y+3u=5$ (3)	$-y+3z=7$ (6)		
$x-y-2z+u=-2$ (4)			

OPERATION FOR ELIMINATING x.

$$\begin{array}{ll} \text{Mult. (1) by 2,} & 2x-4y+2z+2u=10 \\ \text{Writing (3),} & \underline{2x-4y+3u=5} \\ \text{Subtracting,} & 2z-u=5 \;(5) \end{array}$$

$$\begin{array}{ll} \text{Writing (1),} & x-2y+z+u=5 \\ \text{and (4),} & \underline{x-y-2z+u=-2} \\ \text{Subtracting,} & -y+3z=7 \;(6) \end{array}$$

OPERATION FOR ELIMINATING y.

$$\begin{array}{ll} \text{Writing (2),} & 3y-2z+2u=2 \\ \text{Mult. (6) by 3,} & \underline{-3y+9z=21} \\ \text{Adding,} & 7z+2u=23 \;(7) \end{array}$$

OPERATION FOR ELIMINATING u.

$$\begin{array}{ll} \text{Mult. (5) by 2,} & 4z-2u=10 \\ \text{Writing (7),} & \underline{7z+2u=23} \\ \text{Adding,} & 11z=33 \;(8) \end{array}$$

From (1), By Substitution.	From (6), By Substitution.	From (5), By Substitution.	From (8), By Division.
$x-4+3+1=5$,	$-y+9=7$,	$6-u=5$,	$z=3$.
$x=5$.	$y=2$.	$u=1$.	

EXPLANATION.—For convenience we arrange the similar terms under each other, as shown in the first set of equations. We select x to be eliminated first, because it occurs in only three of the equations, and its coefficients are small and easily made alike. We combine (1) and (3), obtaining (5), which we place in the second set. Equation (2) does not contain x, so it is placed immediately in the same set. We then combine (1) and (4), taking these because in them the coefficients of x are alike, obtaining (6), which completes the second set. We have now three equations, containing only three of the unknown quantities. To form these, all the original set were used, as shown by the connecting lines in the operation.

In a similar way we eliminate y from this set of equations, and obtain the third set, *two* equations containing *two* unknown quantities. These combined give *one* equation containing only *one* unknown quantity.

The elimination is now completed. By division we obtain the value of z. The values of the other letters are obtained by successive substitutions.

MENTAL EXERCISE ON THE FOREGOING SOLUTION.

1. In the foregoing operation why were equations (1) and (4) combined, instead of (3) and (4)?

2. If z were selected to be eliminated from the first set of equations, how should the equations be taken so as to give the least work?

3. If y were selected to be eliminated first, which equations would it be best to combine?

4. In the second set, what quantity might have been eliminated as easily as y?

5. Why was u selected to be eliminated from the third set?

6. Why was equation (6) selected for finding the value of y by substitution?

7. Could more than three equations not containing x have been derived from the first set?

Solve the following sets of equations:

WRITTEN EXAMPLES.

$$1.\ \left\{\begin{aligned} x - 3y + 2z - u &= 6 \\ 3x + 2y - z + 3u &= 13 \\ 2x - 2y + 2z - 2u &= 8 \\ x + y + z + u &= 10 \end{aligned}\right\}.$$

$$2.\ \left\{\begin{aligned} 3x + 2y - 4z &= 11 \\ 2x + y + u &= 16 \\ x + z + u &= 10 \\ 2y + z - u &= 9 \end{aligned}\right\}.$$

$$3.\ \left\{\begin{aligned} x + y + z + u &= 14 \\ 3x + 2y - z &= 28 \\ 4x - y &= 29 \\ 3x - z &= 22 \end{aligned}\right\}.$$

$$4.\ \left\{\begin{aligned} 2y + 3z + u &= 15 \\ x + y + z - u &= 8 \\ x + y + z - 2u &= 7 \\ 3x + 2y + 2z &= 21 \end{aligned}\right\}.$$

NOTE.—To solve the following set of equations, add together the corresponding members of the four equations. Divide each member of the resulting equation by 3, and from this result then subtract, member by member, each of the original equations. This will give the values of the letters.

$$5.\ \left\{\begin{aligned} x + y + z &= 13 \\ x + y + u &= 12 \\ x + z + u &= 14 \\ y + z + u &= 9 \end{aligned}\right\}.$$

NOTE.—Solve the following as given above, but divide by 2 instead of by 3:

$$6.\ \left\{\begin{aligned} \frac{1}{x} + \frac{1}{y} &= 6 \\ \frac{1}{x} + \frac{1}{z} &= 8 \\ \frac{1}{y} + \frac{1}{z} &= 10 \end{aligned}\right\}. \qquad 7.\ \left\{\begin{aligned} x + y &= 10 \\ x + z &= 12 \\ y + z &= 14 \end{aligned}\right\}.$$

SECTION LVII.

MENTAL EXERCISE IN THE STATEMENT OF PROBLEMS.

1. In the number 32 there are how many tens, and how many units?

NOTE.—In the number 32 there are 3 tens and 2 units. The 3 tens equal 30, and the 2 units equal 2. The number, therefore, is equal to 30 and 2, or $30 + 2$. If x represent the 3, or the tens figure, and y represent the 2, or the units figure, then there would be x tens, which equals $10x$, and y units, which equal $1y$, or y. Therefore the number would be equal to $10x + y$.

2. If x represents the 10's figure or digit of a certain number, and y represents the units digit, what expression represents the number?

3. If the digits or figures of the number be inverted, so that y stands for the 10's digit and x for the units digit, what expression will represent the number?

4. If $\frac{x}{y}$ be such a fraction that adding 3 to the numerator and x to the denominator makes it equal to $\frac{1}{2}$, what equation can be formed?

5. If a man can do a piece of work in 6 days, what fractional part can he do in 1 day? and what fractional part in 4 days?

6. If a man can do a piece of work in x days, what fractional part can he do in 1 day? What fractional part in 5 days?

7. If a man can do a piece of work in y days, and works 6 days, what fractional part of the work will he do?

8. If one man can do a piece of work in x days, and works 5 days, and another can do it in y days, and works 8 days, what fractional parts will each do?

9. If two men perform a piece of work, one doing the $\frac{6}{x}$ part of it, and the other the $\frac{8}{y}$ part of it, what will the sum of the two fractions $\frac{6}{x}$ and $\frac{8}{y}$ equal?

10. If A can dig a trench in x days, and B in y days, and A works alone 5 days and then calls in B, and the two together finish the work in 7 days more, how many days does each work? What fractional part does each do? What is the sum of the fractional parts equal to?

11. If A can paint a fence in 10 hours and B in 6 hours, and the two together in x hours, what fractional parts can A and B paint in one hour? and what fractional part can both together paint in one hour?

12. If A can mow $\frac{1}{6}$ of a field in a day, and B can mow $\frac{1}{10}$th of it in the same time, and together they can mow $\frac{1}{x}$ part of it in one day, what equation can be made?

SECTION LVIII.

PROBLEMS TO BE SOLVED BY TWO OR MORE UNKNOWN QUANTITIES.

PROBLEMS TO BE SOLVED BY TWO UNKNOWN QUANTITIES.

1. 6 years ago A was $\frac{2}{3}$ as old as B. In 4 years he will be $\frac{4}{5}$ as old. How old is each now?

2. A drover had 52 sheep in two pens. He sold 17, taking $\frac{1}{4}$ of those in one pen, and 4 less than $\frac{1}{2}$ those in the other. How many sheep were in each pen?

3. A man sold 59 cattle, taking from one drove $\frac{5}{8}$ of the number, and from the other $\frac{3}{5}$ of the number. He then sold 15, taking $\frac{1}{3}$ of the first remainder, and $\frac{1}{2}$ of the second. How many were in each drove at first?

4. In a certain number the tens figure, or digit, is 4 times the units digit. If 13 be taken from $\frac{1}{2}$ the number, the result will be a number with the same digits in an inverted order. What is the number?

5. If the numerator of a certain fraction be increased by 1, and the denominator by the numerator, the result will equal $\frac{1}{2}$. But if the numerator be increased by the denominator, and the denominator decreased by 3, the result will equal 3. What is the fraction?

6. A works 10 days at a wall, and then B finishes it in 4 days. If A had worked 6 days, B could have finished the work in 6 days. In how many days could each have done the work?

7. A man and his son can together hoe a field in 6 hours; but the man being called away after working 4 hours, the son finishes it in 5 hours more. In what time can each hoe the field?

8. If 4 be added to the numerator of a fraction, and 9 be added to the denominator, the fraction will equal $\frac{1}{2}$. But if 9 be added to the numerator and 4 to the denominator, the fraction will equal $\frac{4}{3}$. What is the fraction?

9. A number of two digits is 7 times the sum of its digits, and the tens digit is 2 greater than the units digit. What is the number?

10. A number of two digits is 6 more than 5 times the units digit, and if the digits be inverted the resulting number will be 7 times the sum of the digits. What is the number?

11. A and B engage to do a piece of work in 10 days; but

B being sick 5 days, they take 13 days to complete it. In what time could each have done it?

12. A farmer had sheep in two fields. Desiring to give them more room, he placed in a third field 16 sheep from one of the fields, and $\frac{1}{4}$ of those in the other. The three fields now contained equal numbers of sheep. How many were in each of the two fields at first?

13. A number of two digits is 4 times the sum of the digits, and if 9 be added to the number the result will have its digits in an inverted order. What is the number?

14. A farmer has some sheep and oxen. Allowing 3 sheep to an acre of ground, and 1 ox to an acre, it would take a field of 16 acres to accommodate them. But having sold 12 sheep and bought enough oxen to double their number, he finds that 14 acres will be sufficient. How many of each had he at first?

15. A tank can be filled from two pipes. If both are running, the tank will be filled in 16 hours. If one runs 10 hours alone, the other can fill it in 40 hours. In what time will each fill it?

16. A boy spends his money for tops of a certain kind, purchasing 6. Had he bought others, 1 cent a top less, he could have purchased 8. How much money had he? and what was the cost of the tops which he bought?

17. The tens digit of a certain number is $\frac{1}{5}$ greater than the units digit, and if 9 be subtracted from the number, the result will be a number with the same digits in an inverted order. What is the number?

18. A and B together have \$1400; B gives A \$300, and then A spends $\frac{2}{3}$ of his money, and B spends $\frac{1}{2}$ of his. A has left now \$50 more than B. How much had each at first?

19. A dealer buys 20 chickens and 15 geese for \$20. He sells 10 chickens at an advance of 10 cents apiece, and 12

geese at an advance of 20 cents apiece, receiving for them \$17. How much did he pay for each?

20. A drover bought sheep for \$4 and cows for \$25, spending \$320. 5 sheep and 2 cows having died while being driven to market, he sold the sheep for \$6, and the cows for \$40, making a profit of \$70. How many sheep and cows did he purchase?

PROBLEMS TO BE SOLVED BY MORE THAN TWO UNKNOWN QUANTITIES.

1. A person wishes to weigh three packages, but has not weights enough. But he finds that the first will balance the second and a 3-pound weight; that the second will balance the third and a 2-pound weight, and the second and third together will balance the first and a 7-pound weight. How many pounds did each package weigh?

2. Three bags of coffee, taken two at a time, weighed 94, 76, and 90 pounds. How much did each weigh?

3. The number of bones in the fingers exceeds the number in the wrist by 6, the number in the wrist is twice as many as in the hand, less 2, and the number in the wrist and hand together is 1 less than in the fingers. How many bones are in each?

4. If to twice the height of Bunker Hill Monument 40 feet be added, the sum will be equal to the height of the pyramid of Cheops. If the monument, starting at one angle, be laid three times along the side of the base of the pyramid, there will be left of the side 6 feet less than half the length of the monument. If the monument be put on top of the pyramid, the total height would be less than a side of the base of the pyramid by 64 feet. What are the dimensions?

5. A, B, and C together can dig a trench in 5 days; A and C together can dig it in $7\frac{1}{2}$ days. A and B work at it 4 days,

when C takes B's place, and A and C finish the work in 3 more days. In what time could each have done it?

6. A man and his two sons set up a fence in 21 days. The man and oldest son could have done it in 30 days, and the two sons together could have done it in 42 days. In how many days could each have done it?

7. The sum of the three digits of a certain number is 18. If 36 be added to twice the number, the result will be a number with the same digits in an inverted order. The units digit is equal to the sum of the tens and hundreds digits. What is the number?

SUGGESTION.—Let x = hundreds digit, y = tens digit, and z = units digit. Thus, $100x + 10y + z$ = the number.

8. A comparison of the cubic inches in the brains of the European race, of the Bushmen of Africa and of the gorilla gives the following results: The European's brain contains 7 cubic inches more than 3 times the gorilla's. The Bushman's contains 30 more than half the European's. The Bushman's and gorilla's together are 12 more than the European's. How many cubic inches in the brain of each?

9. A, B, and C have altogether $1600. They trade separately, and A gains $200, B loses half his money, and C doubles his. They have now between them $1850, A having the same that B had at first. How much did each start with?

10. The sum of two numbers of two digits each is 56. If the digits of the smaller number be inverted, the result will be 10 more than the larger number. The sum of the four digits is 11, and the difference between the tens' digits is 1. What are the numbers?

SECTION LIX.

MISCELLANEOUS PROBLEMS, TO BE SOLVED BY SIMPLE EQUATIONS.

PROBLEM. A man can do a piece of work in 6 days, and his son in 8 days. In how many days will they do it if they both work together?

STATEMENT. Let x = the number of days it would take both together.

Then, $\frac{1}{x}$ = the fractional part they could do in one day;

$\frac{1}{6}$ = the fractional part the man could do in one day, *because he could do it all in 6 days;*

and $\frac{1}{8}$ = the fractional part the son could do in one day, *because he could do it all in 8 days.*

But the parts which each could do separately in one day must together be equal to the part which they would do in one day if they worked together.

Therefore we have the equation, $\frac{1}{6} + \frac{1}{8} = \frac{1}{x}$.

SOLUTION. *Clearing of fractions,* $4x + 3x = 24$.

Collecting, $7x = 24$.

Dividing by 7, $x = 3\frac{3}{7}$, *the number of days it would take both to do the work.*

PROBLEMS.

1. The head of a great northern whale was 2 feet longer than twice the length of the tail. The body was 1 foot less than twice the length of the head and tail together, and the whole whale was 7 feet more than 4 times the length of the

head. What were the lengths of the three parts, and of the whale?

2. A person bought 3 yards of linen and 6 yards of muslin, the linen costing 3 times as much per yard as the muslin. The price of both having risen 2 cents a yard, he found that 2 yards of linen and 8 yards of muslin cost 10 cents more than the former lot. What were the prices at first?

3. A man sells new milk at 4 cents a quart. How many quarts of milk does it take to give one quart of cream, if cream can be sold for 20 cents and skimmed milk for 2?

SUGGESTION—If 1 quart of cream be taken from x quarts of milk, how many quarts of milk will be left?

4. There are two fractions, of which the numerator of the second is equal to the denominator of the first. The product of the two fractions, plus the second, is equal to $\frac{7}{8}$. If the second be inverted and added to the first, the sum will be 2. If 7 be added to the denominator of the second fraction, its value will be $\frac{1}{3}$. What are the fractions?

5. A man placed 54 sheep in 3 adjoining fields, putting in the second field 4 less than in the first. $\frac{1}{4}$ of those in the first field and $\frac{1}{3}$ of those in the third having broken into the second, that field contained half the sheep. How many were in each field at first?

6. From a certain cask there were drawn out 10 gallons of liquid, and as much more put in as was left. There were then drawn out 30 gallons, when there remained only half the amount that was left after the first draught. How many gallons did the cask contain at first?

7. In gunpowder there are equal parts of saltpetre and sulphur, and 4 times as much charcoal as both of the others together. How much of each is there in 10 pounds of powder?

8. A stack of hay will last one horse 6 months, and a cow 4 months. How long will it last 3 horses and 4 cows?

SUGGESTION.—If 1 horse will eat $\frac{1}{6}$ of the stack in one month, what part will 4 horses eat in the same time?

9. On the first voyage of Columbus across the Atlantic he was 12 days longer in his voyage from Spain to the West Indies than he was in returning. He spent at the West Indies 20 days less than twice the time of the return voyage, and the whole expedition covered 224 days. How many days were spent in each of the three parts?

10. A grain-dealer had two kinds of oats. One weighed 32 pounds to the bushel, the other 24 pounds. A customer bought 6 bushels of the better oats; but when they were sent to him he found that they only weighed 30 pounds to the bushel. How many bushels of the lighter oats had been mixed with the others?

11. Two sheep, which weighed together 91 pounds, were fattened. One gained an amount equal to half its weight, and the other gained 10 pounds more than $\frac{1}{3}$ of its weight, when they were found to weigh together 138 pounds. How much did each weigh after being fattened?

12. A man and his son could together row a boat 12 miles in 3 hours. The man rows 1 hour, then both together row 1 hour, and the boy finishes the 12 miles in 4 hours more. How many miles per hour can each row?

13. What are eggs a dozen when 3 more for the price of a dozen would make each egg cost half a cent less?

14. The length of the large suspension-bridge at Niagara is 94 feet less than 3 times the distance from the top of the supporting towers to the water. The distance of the railroad-track from the water is 5 feet more than 4 times the height of the towers above the track. If one tower were laid 13 times along the track, there would be left 41 feet of the bridge. What are the length of the bridge, the distance of the railroad-track from the water, and the height of the towers above the track?

SECTION LX.

REVIEW.

MENTAL EXERCISE.

1. In the expression a^4 what does the exponent 4 show?

2. Multiply $a^4 \times a^4 \times a^4$, taking a^4 3 times. The exponent of the product is how many times the exponent of a^4?

3. Repeat Theorem I.

4. Repeat Theorem II.

Give the squares indicated in the following:

5. $(a+b)^2$.
6. $(a-2b)^2$.
7. $(2a-b)^2$.
8. $(a^2-ax)^2$.
9. $(x-y)^2$.
10. $(4x+2y)^2$.

11. Why are exponents added in multiplication? (Art. 42.)
12. Why are exponents subtracted in division? (Art 67.)
13. What is a Binomial? Give an example.
14. What is a Monomial? Give an example.
15. What is a Trinomial? Give an example.
16. $a^2+2ab+b^2$ is the square of what quantity?
17. $16x^2+24xy+9y^2$ is the square of what?
18. Is $25x^2-10xy+16y^2$ a perfect square?
19. Perform the following multiplications:

2×3, $\quad 2\times -3$, $\quad -2\times 3$, $\quad -2\times -3$.

20. Where the signs of the quantities are alike, as in the first and last cases, what is the sign of the product?

21. Where the signs are unlike, as in the second and third cases, what is the sign of the product?

From these cases we derive the following important

Rule.*—In multiplication, and also in division, like signs produce +, and unlike signs produce −.*

PART III.

POWERS, ROOTS AND QUADRATIC EQUATIONS.

SECTION LXI.

INVOLUTION OF MONOMIALS.

129. We have seen that x^6 means the product of 6 x's, and that $(a + x)^2$ means the product of $a + x$ and $a + x$. The exponent shows how many times the letter, or quantity, is taken as a factor.

If a quantity is taken any number of times as a factor, the result is called a **Power** of that quantity.

Powers are called *first power*, second power, etc., according to the number of times the quantity is taken as a factor. Thus, a^3 is the *third power* of a, because the exponent 3 indicates that a is taken 3 times as a factor. 25 is the second power of 5, because the factors of 25 are 5 and 5. $(a + x)^4$ indicates the 4th power of $(a + x)$.

The first power of any quantity is the quantity itself. Thus,

$$a = a^1, \quad 5 = 5^1, \quad a + x = (a + x)^1.$$

The **second power** is usually called the **Square**.

The **third power** is often called the **Cube**.

The **higher powers** are distinguished by the ordinal adjectives, **fourth, fifth,** etc.

130. Involution is the process of raising a quantity to any given power.

To **involve** a quantity is to raise it to a given power.

Any power may be indicated by affixing to the quantity an exponent equal to the number indicated by the power.

Any power of a quantity may be found by multiplying the quantity together the number of times indicated by the exponent.

MENTAL EXERCISE.

1. What is the fifth power of a?
2. What is the second power of 7?
3. What is the third power of 3?
4. What is the third power of x^2?
5. What is the second power of a^3x?
6. What is the third power of c^2d^3?

131. If we wish to find any power of a monomial composed of letters, it can be done more shortly than by continued multiplication.

Ex. What is the fourth power of a^3?

SOLUTION. We may represent the continued multiplication thus: $a^3a^3a^3a^3$, which equals a^{12}. That is, the exponent of the power, a^{12}, contains the exponent of a^3 as many times as a^3 was to be taken as a factor, or four times.

Rule.—To raise a monomial to any power, multiply each exponent by the number which indicates the power.

An expression like $(c^2dy^3)^4$ means that c^2dy^3 is to be raised to its 4th power. The result is $c^8d^4y^{12}$. The original exponent of d was 1, understood.

Raise the following monomials to the indicated power:

MENTAL EXERCISE.

1. $(a^2b)^3$.
2. $(a^2x^3y)^4$.
3. $(c^2d^5x)^7$.
4. $(a^4y^2z)^5$.
5. $(x^3y^2)^4$.
6. $(ac^2d^3x^4y^5)^3$.
7. $(cm^4nx^5)^2$.
8. $(b^2dm^{10}x)^3$.
9. $(a^2c^4x^8y^2)^4$.

132. The successive powers of $-a$ are—

1st power, $-a = -a$.
2d power, $-a \times -a = +a^2$.
3d power, $a^2 \times -a = -a^3$.
4th power, $-a^3 \times -a = +a^4$.
5th power, $a^4 \times -a = -a^5$.
6th power, $-a^5 \times -a = +a^6$.

We notice that the first, third, and fifth powers, which are the odd-numbered ones, have the $-$ sign; while the second, fourth, and sixth powers, which are even numbered, have the $+$ sign.

This also follows from the principle of multiplication, that *negative* quantities taken two and two give *positive* products.

For powers of Monomials, we have the general

***Rule.**—I. If the monomial has a coefficient, raise it to the required power by ordinary multiplication.*

II. Multiply the exponent of each letter by the exponent of the required power.

III. If the sign of the monomial is $+$, the sign of any power will be $+$. But if the sign is $-$, the even powers will have the sign $+$, and the odd powers will have the sign $-$.

Give the powers indicated in the following expressions:

MENTAL EXERCISE.

1. $(3x^3y^2)^2$.
2. $(-5m^3n^5)^3$.
3. $(2a^3c^5)^3$.
4. $(-2x^4z^9)^4$.
5. $(6ax^3y)^2$.
6. $(-xy^2z^4)^3$.
7. $(-10a^2x^4y^3)^4$.
8. $(-4m^4n^5y^2)^2$.
9. $(9a^3cx^2)^2$.

Expand the following quantities; that is, perform the indicated involution and multiplication:

MENTAL EXERCISE.

1. $3(a^2x)^2$.
2. $4a(a^2y^2)^3$.
3. $3(a^2)^2(x^2)^3$.
4. $4(3a)^2(2a)$.
5. $2(3x^2)^3(4xy)$.
6. $5(a^2x)^5(ax^2)^2$.
7. $(2x^2y^2)^2(3a^2b^2)^2$.
8. $2(x^2y)^3(2xy^2)$.

133. Since, in multiplying fractions together, we multiply numerator by numerator and denominator by denominator, in raising a fraction to any power we raise the numerator and denominator separately to that power. Thus, $\left(\frac{a}{x}\right)^3 = \frac{a^3}{x^3}$, for $\left(\frac{a}{x}\right)^3 = \frac{a}{x} \times \frac{a}{x} \times \frac{a}{x} = \frac{a^3}{x^3}$.

Raise the following fractions to the indicated powers:

MENTAL EXERCISE.

1. $\left(\frac{a^2}{2b}\right)^2$.
2. $\left(-\frac{x^3y}{3ab^2}\right)^3$
3. $\left(-\frac{4b^2x}{ay^3}\right)^2$.
4. $\left(\frac{ax^2y}{m^2z^4}\right)^4$.
5. $\left(-\frac{c^2d^3}{xy^4}\right)^4$.
6. $\left(-\frac{a^3x}{b^2d^4y}\right)^4$.
7. $\left(\frac{2x^3}{3a^2y^4}\right)^2$.
8. $\left(\frac{7ax^3}{ac^2d}\right)^2$.
9. $\left(\frac{1}{3ax^3}\right)^3$.

SECTION LXII.

INVOLUTION OF POLYNOMIALS.

Raise the following binomials to the indicated second power by the use of Theorems I. and II.:

1. $(a+b)^2$.
2. $(a-x)^2$.
3. $(m+n)^2$.
4. $(x-2y)^2$.
5. $(4+x)^2$.
6. $(3m+2n)^2$.
7. $(5a-6b)^2$.
8. $(3ax+2x^2)^2$.
9. $(4-3y)^2$.
10. $(2x-x^2)^2$.
11. $(5y^3-2y^2)^2$.
12. $(3x^2+y)^2$.

134. By actual multiplication, polynomials may be raised to any given integral power.

Ex. Raise $a+x$ to the third power.

OPERATION.

$$\begin{array}{llll}
a + x & & & \\
\underline{a + x} & & & \\
a^2 + ax & & & \\
\underline{\phantom{a^2 +{}} ax + x^2} & & & \\
a^2 + 2ax + x^2 & & & \\
\underline{a + x} & & & \\
a^3 + 2a^2x + ax^2 & & & \\
\underline{\phantom{a^3 +{}} a^2x + 2ax^2 + x^3} & & & \\
a^3 + 3a^2x + 3ax^2 + x^3 & & &
\end{array}$$

Raise by multiplication the following binomials to the indicated powers:

WRITTEN EXAMPLES.

1. $(x-y)^3$.
2. $(3a-2b)^3$.
3. $(a+2x)^4$.
4. $(4x-5y)^3$.

SECTION LXIII.

BINOMIAL THEOREM.

135. The Binomial Theorem gives a method of finding powers of binomials without the continued multiplication. It was discovered by Sir Isaac Newton.

We find the following powers by continued multiplication:

$$(x + y)^1 = x + y.$$
$$(x + y)^2 = x^2 + 2xy + y^2.$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + x^3.$$
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$
$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

$$(x - y)^1 = x - y.$$
$$(x - y)^2 = x^2 - 2xy + y^2.$$
$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3.$$
$$(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4.$$
$$(x - y)^5 = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5.$$

136. From an examination of these powers of $x + y$ and $x - y$, we derive the following

PRINCIPLES OF THE BINOMIAL THEOREM.

I. *Of Signs.*—If the connecting sign in the binomial is +, as in $x + y$, all the signs of the power are +.

If the connecting sign is −, as in $x - y$, every alternate sign is −, beginning with the sign of the second term.

II. *Of Coefficients.*—The coefficient of the first term is always 1; of the second term it is the same as the exponent of the power to which the binomial is to be raised. Thus, in the 4th power of $x + y$ the coefficient of the second term is 4; in the

5th power it is 5. The coefficient of the third term may be found by multiplying the coefficient of the second term by the exponent of the first letter in that term, and dividing the result by 2, the number which marks the place of the second term. And, in the same way, to find the coefficient of any term, multiply the coefficient of the preceding term by the exponent of the first letter in that term, and divide by the number which marks its place. Thus, if the fourth term be $10\,x^2y^4$, we find the coefficient of the fifth term by multiplying 10 by 2, and dividing by 4, which gives 5.

III. *Of Letters and their Exponents.*—The first letter, x, occurs in the first term, with an exponent equal to the exponent of the power to which the binomial is to be raised. It occurs in the succeeding terms, except the last, with an exponent successively decreasing by 1, till in the next to the last term it is 1. The second letter occurs in the second term, with an exponent 1, and in all the succeeding terms with an exponent successively increased by 1, till in the last term it is equal to the exponent of the power of the binomial.

137. In using these principles to find any power of a binomial, we do not find what all the signs will be, then what will be all the literal parts of the terms, and then insert the coefficients, but by the use of the proper principles we put together one term after another in order, putting down first the first term, then the sign, coefficient, and literal parts of the second in order, and so on, till the whole power is completed.

This is illustrated in the solution of the following:

Ex. Raise $x - y$ to the 7th power.

Solution. We first determine the first term. From Prin. I. its sign is +, and from Prin. II. its coefficient is 1, and from

Prin. III. x is the only letter, and is raised to its 7th power. Hence, the first term is x^7.

The connecting sign of the binomial being $-$, the sign of the second term, according to Prin. I., is $-$. According to Prin. II., the coefficient is equal to the power of the binomial, or 7. According to Prin. III., the letter x has an exponent one less than its former exponent, or 6, and y occurs with an exponent 1; hence, the second term is $-7x^6y$, and the power as thus far constructed is

$$\begin{array}{ll} x^7 - 7x^6y & \\ \quad\ \ 6 & \\ 2\overline{)42} & \\ \quad 21, & \text{3d coefficient.} \end{array}$$

The next sign is $+$. The next coefficient is found by taking 6 times 7, which equals 42, and dividing by 2, which gives 21. The literal part is x^5y^2, the exponent of x being diminished 1, and that of y being as much increased. The third term, therefore, is $21x^5y^2$, and the work stands thus:

$$\begin{array}{ccc} x^7 - 7x^6y & + \; 21x^5y^2 & \\ 6 & 5 & \\ 2\overline{)42} & 3\overline{)105} & \\ 21 & 35, & \text{4th coefficient.} \end{array}$$

The next sign is $-$. The coefficient is $21 \times 5 \div 3 = 35$. The literal parts are x^4 and y^3. The whole term is $35x^4y^3$, and the work stands thus:

$$\begin{array}{cccc} x^7 - 7x^6y & + \; 21x^5y^2 & - \; 35x^4y^3 & \\ 6 & 5 & 4 & \\ 2\overline{)42} & 3\overline{)105} & 4\overline{)140} & \\ 21 & 35 & 35, & \text{5th coefficient.} \end{array}$$

Proceeding in this way, by successive applications of Principles I., II., and III., the completed work would stand—

$$x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$$

$$\begin{array}{cccccc} \underline{6} & \underline{5} & \underline{4} & \underline{3} & \underline{2} & \underline{1} \\ 2\overline{)42} & 3\overline{)105} & 4\overline{)140} & 5\overline{)105} & 6\overline{)42} & 7\overline{)7} \\ \hline 21 & 35 & 35 & 21 & 7 & 1 \end{array}$$

NOTE.—The work for the coefficients can generally be done mentally; but the method given above is very convenient, and serves to fix the operation in the mind.

Rule.—To raise a binomial to any required power, we construct each term of the power separately, and in regular order, determining the **sign, coefficient,** *and* **literal part** *by the successive use of Principles I., II., and III. of the Binomial Theorem.*

138. The terms of the power always number one more than the exponent of the binomial.

The coefficients are the same at equal distances from each end; hence they need only be calculated as far as to the middle of the terms.

The sum of the exponents of the letters in each term is always the same and is equal to the exponent of the binomial.

Raise the following binomials to the indicated powers by means of the Binomial Theorem:

WRITTEN EXAMPLES.

1. $(x + y)^8$.
2. $(a - x)^5$.
3. $(c + d)^7$.
4. $(m - n)^8$.
5. $(b + c)^6$.
6. $(a + c)^5$.
7. $(c + d)^8$.
8. $(m + n)^6$.
9. $(a - x)^4$.

MENTAL EXERCISE.

1. $(x - y)^3$.
2. $(a + b)^4$.
3. $(c + d)^3$.
4. $(x + y)^5$.
5. $(m - n)^4$.
6. $(a - b)^5$.

139. Ex. Raise $4a^2 + 3x$ to the 4th power.

SOLUTION. The first letter has already both a coefficient and an exponent, and the second has a coefficient, which are greater than 1; we cannot therefore treat the letters as we have done in previous examples. If, however, we enclose each term in a parenthesis, thus, $(4a^2) + (3x)$, and treat each parenthesis as though it were a letter, we can proceed as before. Thus,

$$\big((4a^2) + (3x)\big)^4 = (4a^2)^4 + 4(4a^2)^3(3x) + 6(4a^2)^2(3x)^2$$
$$+ 4(4a^2)(3x)^3 + (3x)^4.$$

Expanding each term, we have

$$256a^8 + 768a^6x + 864a^4x^2 + 432a^2x^3 + 81x^4.$$

Raise the following binomials to the indicated powers by means of the Binomial Theorem:

WRITTEN EXAMPLES.

1. $(2a + b)^8$.
2. $(3x - y)^4$.
3. $(c + 2d)^5$.
4. $(x^3 + y^2)^6$.
5. $(4y + z)^4$.
6. $(y^2 + y)^4$.
7. $(\frac{1}{2}x^2 - 2x)^3$.
8. $(a^2x + x)^4$.
9. $(x + \frac{1}{4}y)^5$.

The cube of $a + b$ is $a^3 + 3a^2b + 3ab^2 + b^3$. That is, *the cube of the sum of two quantities is equal to the cube of the first, plus 3 times the product of the square of the first, and the second, plus 3 times the product of the first and the square of the second, plus the cube of the second.*

If the cube of the difference of two quantities be taken, the result will be the same, except that the signs of the second and fourth terms will be minus.

Give the following indicated cubes:

MENTAL EXERCISE.

1. $(x + y)^3$
2. $(a - x)^3$.
3. $(m + n)^3$.

SECTION LXIV.

EVOLUTION OF MONOMIALS.

140. WHAT is the quantity whose second power, or square, is $9x^6$?

SOLUTION. We know that 9 is the square of 3, for $3 \times 3 = 9$. If x^6 be the square of some power of x, the original exponent was multiplied by 2 to get x^6 (Art. 131). Therefore we divide the present exponent by 2 to find the original one. The result is x^3. Hence, $3x^3$ is the quantity whose second power, or square, is $9x^6$. It is called the second, or square, root of $9x^6$.

Ex. What is the square root of $16x^4y^{10}$?

SOLUTION. We wish to find the quantity whose second power is $16x^4y^{10}$. We know that the second power of 4 is 16. The exponents of x^4y^{10} must be divided by 2 to find those exponents which, multiplied by 2, will be 4 and 10. Hence, the quantity is $4x^2y^5$.

Ex. What is the quantity whose third power, or cube, is $27c^6d^3$?

SOLUTION. We know that 27 is the cube of 3, so 3 is the coefficient of that quantity. And since the exponents in the quantity were multiplied by 3 to get c^6d^3, we must divide by 3 to find what they were before. Hence, the quantity is $3c^2d$.

Since the cube of $3c^2d$ is $27c^6d^3$, $3c^2d$ is called the cube root of $27c^6d^3$.

141. If one quantity be a certain *power* of another, the second is the same *root* of the first.

Since any power of a quantity is composed of that quantity taken a certain number of times as a factor, the factors are all equal to each other. Therefore a root of a quantity is one of its equal factors, the number of the root being the same as the number of the equal factors of which it is one.

Ex. What is the fourth root of a^8b^{12}?

Solution. Since the exponents of the desired quantity were multiplied by 4 to get a^8b^{12}, we divide these exponents by 4 to find those of the fourth root. This gives us a^2b^3.

***Rule.**—To take any root of a monomial, take the root of the coefficient, as in Arithmetic, and divide the exponents of the letters by the number which indicates the root.*

142. The student should make himself familiar with the following

Table of Powers of Numbers.

The *Square* of
$2 = 4$,
$3 = 9$,
$4 = 16$,
$5 = 25$,
$6 = 36$,
$7 = 49$,
$8 = 64$,
$9 = 81$,
$10 = 100$,
$11 = 121$,
$12 = 144$.

The *Cube* of
$2 = 8$,
$3 = 27$,
$4 = 64$,
$5 = 125$.

The *Fourth Power* of
$2 = 16$,
$3 = 81$,
$4 = 256$.

The *Fifth Power* of
$2 = 32$,
$3 = 243$.

The *Sixth Power* of
$2 = 64$.

MENTAL EXERCISE.

1. What is the number whose cube is 27?
2. What is the cube root of 64?
3. What is the square root of $49x^4y^2$?
4. What is the fifth root of $32a^{10}c^5$?

5. What is the quantity whose square is $64\,a^4m^2x^8$?
6. What is the square root of $81\,x^2y^6$?
7. What is the 6th root of $c^{12}d^{36}$?
8. What is the quantity whose 4th power is $81\,b^4x^8y^{12}$?

143. The square of x is x^2.
The square of $-x$ is x^2.

That is, x^2 is the square of both $+x$ and $-x$. When it is asked, What is the quantity whose square is x^2? there are two answers: $+x$ or $-x$. That is, there are two square roots.

The 6th power of x is x^6.
The 6th power of $-x$ is x^6.

Hence, there are two 6th roots of x^6, $+x$ and $-x$. Every quantity has two *even* roots which are alike, except as to their signs.

Instead of writing the two roots separately, it is customary to affix a double sign, $\pm$, called *plus* or *minus*, to the root. Thus, the square root of 4 may be either $+2$ or -2, which we indicate by writing ± 2 (read plus or minus 2). The square root of x^2 is written $\pm x$, and is read $+$ or $-$ x, and means $+x$ or $-x$.

144. The third power of a is a^3.
The third power of $-a$ is $-a^3$.

Since the *odd* power of a + quantity is +, and the *odd* power of a − quantity is −, it must follow that the *odd* root of a + quantity is +, and the *odd* root of a − quantity is −.

145. Since the even powers of both + and − quantities are +, there can be no quantity with the sign − which is an *even* power, and no even root of such a quantity can be extracted.

146. A *Root* is indicated by means of the sign $\sqrt{\ }$, called the **radical sign.** The number of the root is shown by a small figure in it. Thus, $\sqrt[6]{a}$ means the 6th root of a, and is so read. $\sqrt[3]{x}$ means the cube root of x. In indicating the *square* root, the figure 2 is generally omitted from the radical sign. Thus, $\sqrt{9}$ indicates the square root of 9. Sometimes the indicated root can be taken, and sometimes it cannot. Thus, $\sqrt{9} = 3$ and $\sqrt[3]{x^6} = x^2$, but in the expressions $\sqrt{x}$, or $\sqrt[3]{6}$, the indicated root cannot be found.

The indicated root of a quantity is called **a radical quantity**, or simply a **radical.**

The small figure indicating the number, or **degree**, as it is usually called, of the root is called the **index.** The plural of this is **indices.**

147. From these discussions we derive, for the extraction of roots of *monomials*, the following general

Rule.—*I. Find the root of the coefficient as in Arithmetic.*

II. Find the root of the literal part by dividing the exponent of each letter by the index of the root.

III. The **even** *root of any quantity has the double sign* ±. *The* **odd** *root of a positive quantity has the sign* +. *The* **odd** *root of a negative quantity has the sign* −.

Give the roots indicated in the following:

MENTAL EXERCISE.

Note.—Be careful, on taking an even root, to give the sign ±.

1. $\sqrt{9}$.
2. $\sqrt{4a^2}$.
3. $\sqrt[3]{8x^6}$.
4. $\sqrt[4]{x^8y^4}$.
5. $\sqrt[5]{-m^{10}n^{15}}$.
6. $\sqrt[4]{16a^8b^{12}}$.
7. $\sqrt{81a^4x^6y^2}$.
8. $\sqrt[3]{-27c^9d^{12}x^3}$.
9. $\sqrt[5]{32x^{30}y^{25}}$.
10. $\sqrt[3]{8a^{24}b^9c^3}$.

SECTION LXV.

SQUARE ROOT OF POLYNOMIALS.

148. Extract the square root of $x^2 + 2xy + y^2$.

Solution. We can tell at a glance that the required root is $x + y$. But we wish to find some method for getting $x + y$ which we can apply to cases in which we cannot tell at once what the root is. Theorems I. and II. are, *The square of the sum or difference of two quantities is equal to the square of the first,* ± *twice the product of the first and second,* + *the square of the second.* Therefore we first take the square root of the first term to find the first term of the root; and since the second term in the power is twice the product of the first and second quantities, we divide the second term by twice the first root. This gives us the second term of the root.

The work is as follows:

$$
\begin{array}{rl}
 & x^2 + 2xy + y^2 \;\big|\underline{\,x + y\,} \\
 & \underline{x^2 \qquad\qquad\quad} \\
2x + y\,) & \quad 2xy + y^2 \\
 & \quad \underline{2xy + y^2}
\end{array}
$$

In order to see whether the root comes out exactly, we add the second quantity, y, to the trial divisor $2x$, and multiply the whole divisor thus formed by y, and subtract from the first remainder.

The same operation may be continued if the root has more than two terms.

149. The following rule, derived from the above work, is very nearly like the ordinary rule for finding the square root of numbers:

Rule.—*I. Arrange the terms regularly with reference to the powers of one of the letters, as for division.*

II. Take the square root of the first term, for the first term of the root. Square this, and subtract from the quantity.

III. Take twice the root already found for a trial divisor, divide its first term into the first term of the remainder, and write the result as the next term of the root. Annex it also to the trial divisor; multiply the completed divisor by the last term of the root, and subtract from the last remainder.

IV. Repeat the preceding operation till there is no remainder, or a remainder into the first term of which the first term of the trial divisor is not exactly contained; in the latter case there is no exact root.

Ex. Extract the square root of $x^4 - 4x^3 + 10x^2 - 12x + 9$.

OPERATION.

$$\begin{array}{r|l}
 & x^4 - 4x^3 + 10x^2 - 12x + 9 \quad \underline{\,x^2 - 2x + 3\,}\ \text{Root.} \\
 & \underline{x^4 \qquad\qquad\qquad\qquad\qquad\qquad} \\
2x^2 - 2x\,) & -4x^3 + 10x^2 - 12x + 9 \\
 & \underline{-4x^3 + 4x^2 \qquad\qquad\quad} \\
2x^2 - 4x + 3\,) & \qquad\quad 6x^2 - 12x + 9 \\
 & \qquad\quad \underline{6x^2 - 12x + 9}
\end{array}$$

Extract the square root of the following polynomials:

WRITTEN EXAMPLES.

1. $x^2 - 2xy + y^2$.
2. $9a^2 + 24a + 16$.
3. $c^2 - 10cd + 25d^2$.
4. $a^2x^2 - 4ax + 4$.
5. $a^6 + 2a^5 + 3a^4 + 2a^3 + a^2$.
6. $x^4 + 4x^3 + 2x^2 - 4x + 1$.
7. $a^4 - 4a^3 + 4a^2 + 2a^2x - 4ax + x^2$.
8. $b^6 - 4b^4 + 6b^3 + 4b^2 - 12b + 9$.
9. $a^4 - 2a^3x - a^2x^2 + 2ax^3 + x^4$.
10. $25x^4 - 20x^3 + 34x^2 - 12x + 9$.

NOTE.—Be careful to arrange correctly the terms of the following polynomials:

11. $x^2 + y^2 + z^2 + 2xy - 2xz - 2yz.$
12. $c^2 + d^2 + m^2 + 2cd + 2dm + 2cm.$
13. $a^4 + 7a^2x^2 + x^4 - 6a^3x + 6ax^3.$
14. $a^2x^2 + 2a^2xy + a^2y^2 + 2axy^2 + 2ay^3 + y^4.$
15. $x^6 - 4x^5 + 10x^4 - 16x^3 + 17x^2 - 12x + 4.$

SECTION LXVI.

PURE QUADRATIC EQUATIONS.

150. EQUATIONS, any term of which contains the square of one unknown quantity or the product of two unknown quantities, are called equations of the **second degree**, or **quadratic equations.**

Thus, $3x^2 - xy = 11$, $5x^2 = 45$, and $x^2 - 6x = 7$ are quadratic equations.

Quadratic equations like $3x^2 - xy = 11$ and $5x^2 = 45$, in which all the terms containing unknown quantities are of the second degree, are called **incomplete**, or **pure, quadratic equations.**

151. Given $x^2 = 16$, to find the value of x.

If two quantities are equal, their square roots must also be equal. Therefore the square root of x^2 must equal the square root of 16. Now, we found in Art. 143 that the square root of any quantity might have either a + or a − sign, and therefore the double sign ± was affixed to square roots. This would give us $\pm x = \pm 4$ (read plus or minus x equals plus or minus 4).

As we only care to know the value of $+x$, or x, we do not use the sign $-$ as applied to the x, and simply write

$$x = \pm 4,$$

which is the same as $x = +4,$

$$x = -4.$$

Proof. Substituting $+4$ in the first equation, we have

$$16 = 16.$$

Substituting -4, we have $16 = 16.$

Ex. Given $4x^2 = 2x^2 + 18$, to find x.

Transposing, we have $4x^2 - 2x^2 = 18.$

Collecting, $2x^2 = 18.$

Dividing by 2, $x^2 = 9.$

Taking the square root of both members, we find

$$x = \pm 3.$$

Proof. Substituting in the first equation, we have

$$36 = 18 + 18.$$

***Rule.**—Find the value of the square of the unknown quantity, and then extract the square root of both members.*

NOTE.—Remember to use the sign ± when extracting the square root of the second members.

Solve the following equations:

MENTAL EXERCISE.

1. $x^2 = 25.$
2. $2y^2 = 8.$
3. $5y^2 = 45.$
4. $3x^2 = 300.$
5. $6x^2 = 6.$
6. $2x^2 = 32.$

WRITTEN EXAMPLES.

1. $2x^2 = 125 - 3x^2.$
2. $10 + 5x^2 = 4x^2 + 110.$
3. $7x^2 + 2 = 5x^2 + 100.$
4. $5x^2 - 22 = 3x^2 + 140.$

Note.—In the following example the term $4y$ is contained in both members of the equation; it may be cancelled out in each, leaving $3y^2 = 20 - 2y^2$.

5. $3y^2 + 4y = 20 - 2y^2 + 4y.$

6. $2y^2 - 27 = 73 - 2y^2.$

7. $x^2 - 6x = 3(9 - 2x) + 22.$

8. $2x(3x - 12) = x(4x - 24) + 200.$

9. $(x + 2)(x + 3) = 42 + 5x.$

10. $x(2x + 4) - 28 = 100 + 4x.$

11. $x = \dfrac{36}{x}.$

12. $4x + 2 = \dfrac{2x + 16}{x}.$

13. $5y = \dfrac{135 - 20y}{3y - 4}.$

14. $(x + 2)(x - 3) + 2x(x - 3) = (x - 5)(x - 2) + 56.$

15. $\dfrac{(x + 4)(x + 3) - 7x}{2x + 6} = 2x - 6.$

16. $(2x - 3)^2 = (x - 8)(x - 4) + 4.$

17. $(x + 4)(x + 2) = 3(x + 1)^2 - 3.$

18. $\dfrac{x}{x + 9} = \dfrac{4}{x + 4}.$

19. $\dfrac{x + 8}{2x + 3} = \dfrac{3}{x - 2}.$

20. $\dfrac{3}{1 + x} + \dfrac{3}{1 - x} = 8.$

21. $\dfrac{x}{x + 1} + \dfrac{x}{x + 4} = 1.$

22. $\dfrac{4}{x - 3} - \dfrac{4}{x + 3} = \dfrac{1}{3}.$

23. $\dfrac{x}{4} + \dfrac{4}{x} = \dfrac{x}{9} + \dfrac{9}{x}.$

SECTION LXVII.

SOLUTION OF EQUATIONS CONTAINING THE DOUBLE SIGN.

152. EQUATIONS sometimes occur in the following form, in which the coefficients of both x and y are equal.

These can be very conveniently solved by both addition and subtraction:

$$\begin{aligned} x + y &= 9 \\ x - y &= 3 \end{aligned}$$

Adding, $2x = 12$

Dividing by 2, we find $x = 6$

Subtracting the members of the first two equations, $2y = 6$

Dividing by 2, we find $y = 3$

Proof. $6 + 3 = 9,$
$6 - 3 = 3.$

It will be noticed that in the above equations the value of x equals one half the *sum* of the numbers in the second members of the equation, and that y equals one half the *difference* of these numbers.

Give the values of x and y in the following pairs of equations:

MENTAL EXERCISE.

1. $\begin{Bmatrix} x + y = 12 \\ x - y = 2 \end{Bmatrix}.$

2. $\begin{Bmatrix} x + y = 13 \\ x - y = 3 \end{Bmatrix}.$

3. $\begin{Bmatrix} x + 2y = 9 \\ x - 2y = 5 \end{Bmatrix}.$

4. $\begin{Bmatrix} 3x + y = 11 \\ 3x - y = 7 \end{Bmatrix}.$

153. Equations sometimes reduce to the following form:

$$\begin{aligned} x + y &= \pm 6, \\ x - y &= 2. \end{aligned}$$

SOLUTION. The first equation is read $x + y$ equals plus or minus 6. It really stands for two equations,

$$x + y = 6 \text{ and } x + y = -6.$$

Taking $x - y = 2$ with each of these equations gives two sets of equations, which may be solved separately or at the same time. This is shown in the following operation:

	Taking the sign +.	Taking the sign −.	Combining the two operations in one.
	$\left.\begin{aligned} x + y &= 6 \\ x - y &= 2 \end{aligned}\right\}$	$\left.\begin{aligned} x + y &= -6 \\ x - y &= 2 \end{aligned}\right\}$	$\left.\begin{aligned} x + y &= \pm 6 \\ x - y &= 2 \end{aligned}\right\}.$
Adding,	$2x = 8$	$2x = -4$	$2x = 8, \text{ or } -4.$
Dividing by 2,	$x = 4$	$x = -2$	$x = 4, \text{ or } -2.$
Subtracting,	$2y = 4$	$2y = -8$	$2y = 4, \text{ or } -8.$
Dividing by 2,	$y = 2$	$y = -4$	$y = 2, \text{ or } -4.$
The two pairs of values are:	$\left.\begin{aligned} x &= 4 \\ y &= 2 \end{aligned}\right\}$	or $\left.\begin{aligned} x &= -2 \\ y &= -4 \end{aligned}\right\}$	$\left.\begin{aligned} x &= 4, \text{ or } -2 \\ y &= 2, \text{ or } -4 \end{aligned}\right\}.$
Proof.	$4 + 2 = 6$ $4 - 2 = 2$	$-2 - 4 = -6$ $-2 + 4 = 2$	

The third column shows the method usually followed. The meaning of the double operations there performed can easily be seen by comparing it with the other two columns.

Since in the first columns we found $x = 4$ and $y = 2$, and in the second $x = -2$ and $y = -4$, the values in the third column must be taken in the same way. That is, we have $x = 4$ and $y = 2$, or $x = -2$ and $y = -4$. The first values, which result from taking the + part of the double sign, make one set, and the second values, which result from using the − sign, make the other. If the values be taken $x = 4$ and $y = -4$, they will not prove when substituted in the original equations.

Solve the following pairs of equations:

MENTAL AND WRITTEN EXAMPLES.

1. $\left\{\begin{array}{l} x + y = \pm 11 \\ x - y = 3 \end{array}\right\}.$

2. $\left\{\begin{array}{l} x + y = 8 \\ x - y = \pm 4 \end{array}\right\}.$

3. $\left\{\begin{array}{l} x + y = \pm 12 \\ x - y = 8 \end{array}\right\}.$

4. $\left\{\begin{array}{l} x + 3y = 15 \\ x - 3y = \pm 3 \end{array}\right\}.$

154. Sometimes the sign ± is found in both equations.

Ex. Solve

$$\begin{array}{l} x + y = \pm 10, \\ x - y = \pm 4. \end{array}$$

SOLUTION. As either the + or − sign in each equation may be used, there are four ways in which they may be taken:

$$\left.\begin{array}{l} x + y = +10 \\ x - y = +4 \end{array}\right\} \text{ or } \left.\begin{array}{r} -10 \\ -4 \end{array}\right\} \text{ or } \left.\begin{array}{r} +10 \\ -4 \end{array}\right\} \text{ or } \left.\begin{array}{r} -10 \\ +4 \end{array}\right\},$$

Being all expressed by $\left.\begin{array}{l} x + y = \pm 10 \\ x - y = \pm 4 \end{array}\right\}.$

Adding, $2x = 14$ or -14 or 6 or -6,

Dividing by 2, $x = 7$ or -7 or 3 or -3,

Being all expressed by $2x = \pm 14$ or ± 6,

$x = \pm 7$ or ± 3.

Subtracting, $2y = 6$ or -6 or 14 or -4,

Dividing by 2, $y = 3$ or -3 or 7 or -7,

Being all expressed by $2y = \pm 6$ or ± 14,

$y = \pm 3$ or ± 7.

The four pairs of values are—

$$\text{and } \left.\begin{array}{l} x = 7 \\ y = 3 \end{array}\right\} \text{ or } \left.\begin{array}{l} x = -7 \\ y = -3 \end{array}\right\} \text{ or } \left.\begin{array}{l} x = 3 \\ y = 7 \end{array}\right\} \text{ or } \left.\begin{array}{l} x = -3 \\ y = -7 \end{array}\right\},$$

Being all expressed by $\left.\begin{array}{l} x = \pm 7 \text{ or } \pm 3 \\ y = \pm 3 \text{ or } \pm 7 \end{array}\right\}.$

Proof.

$7 + 3 = +10, \quad -7 - 3 = -10, \quad +3 + 7 = +10, \quad -3 - 7 = -10.$

$7 - 3 = +4, \quad -7 + 3 = -4, \quad +3 - 7 = -4, \quad -3 + 7 = +4.$

The last column gives the usual method of work. The others show the meaning of the double signs used in the last, and also how the four pairs of values are to be taken.

It will be observed that the values of x would have been found if, on adding the members of the first two equations, the second members had been first added and then subtracted, and the double sign affixed to both results; and that the value of y would have been found if, on subtracting, the second members had been first subtracted and then added, and the double sign affixed to both results.

Solve the following pairs of equations:

MENTAL AND WRITTEN EXAMPLES.

1. $\left\{\begin{array}{l} x + y = \pm 9 \\ x - y = \pm 3 \end{array}\right\}.$

2. $\left\{\begin{array}{l} x + 2y = \pm 16 \\ x - 2y = \pm 4 \end{array}\right\}.$

3. $\left\{\begin{array}{l} 3x + y = \pm 18 \\ 3x - y = \pm 9 \end{array}\right\}.$

4. $\left\{\begin{array}{l} x + y = \pm 27 \\ x - y = \pm 13 \end{array}\right\}.$

SECTION LXVIII.

PURE QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

155. In solving quadratic equations it is very often necessary to extract the square root of trinomial expressions, like $x^2 + 2xy + y^2$, or $4x^2 + 12x + 9$, which are the squares of binomials.

It is important that the student should be able to recognize trinomials that have exact roots, and to give at once the roots of such as have them.

The use of Theorems I. and II. in factoring has given us the desired method, which is here repeated:

If a trinomial is a perfect square, dividing twice the square root of the first term into the second term, and squaring the result, will give the third term.

The square root of the first term will be the first term of the binomial root; the square root of the third term will be the second term, and the connecting sign will be the same as the sign of the second term of the trinomial.

Ex. Is $a^2 + 4ax + 4x^2$ a perfect square?

Solution. The square root of the first term is a. Twice this, or $2a$, divided into the second term, gives $2x$, and the square of this is $4x^2$, which is the third term. Hence, the trinomial is a perfect square, and is the square of $a + 2x$.

Tell which of the following trinomials are *perfect squares*, and give the square roots of those which are:

MENTAL EXERCISE.

1. $a^2 + 2ab + b^2$.
2. $a^2 + 8a + 16$.
3. $x^2 - 4xy - 4y^2$.
4. $x^2 + 4xy + 4y^2$.
5. $4x^2 + 8xy + 4y^2$.
6. $9x^2 + 12x + 16$.
7. $x^2 - 6x + 9$.
8. $x^2 + 8x + 16$.
9. $x^2 - 18x + 81$.
10. $x^2 - 10x - 25$.
11. $x^2 + 14x + 49$.
12. $x^2 + \frac{1}{2}x + \frac{1}{16}$.
13. $4x^2 + 4xy + y^2$.
14. $4x^2 - 12xy + 9y^2$.
15. $y^2 - 5y + \frac{25}{4}$.
16. $x^2 + x + \frac{1}{4}$.

156. We have before seen how the values of x and y may be found when we know the values of $x + y$ and $x - y$. It is often the shortest way of solving equations to find, first, the values of $x + y$ and $x - y$, and then the values of x and y from these. We will now show some of the methods by which the values of $x + y$ and $x - y$ may be determined from certain kinds of equations.

Ex. If $x^2 + 2xy + y^2 = 49$, what does $x + y$ equal?

SOLUTION. Taking the square root of both members, we have

$$x + y = \pm 7.$$

Ex. If $x^2 - y^2 = 15$,
and $x - y = 3$, what does $x + y$ equal?

SOLUTION. Dividing $x^2 - y^2$ by $x - y$, we have $x + y$, and, dividing 15 by 3, we have 5. But since the divisors and dividends were equal in each division, the quotients must also be equal. Hence, $x + y = 5$. The operation would be called dividing the members of the first equation by those of the second.

This operation may often be used to simplify an equation.

MENTAL EXERCISE.

NOTE.—Be careful to use the double sign on taking the square root.

1. If $x^2 - 2xy + y^2 = 25$, what is the value of $x - y$?
2. If $x^2 + y^2 = 34$,
and $2xy = 30$, what does $x^2 + 2xy + y^2$ equal?
3. If $x^2 + 2xy + y^2 = 49$, what does $x + y$ equal?
4. If $x^2 + y^2 = 34$,
and $2xy = 30$, what does $x^2 - 2xy + y^2$ equal?
5. If $x^2 - 2xy + y^2 = 16$, what does $x - y$ equal?
6. If $x^2 - y^2 = 16$,
and $x + y = 8$, what does $x - y$ equal?
7. If $x^2 + y^2 = 10$,
and $xy = 3$, what does $x^2 + 2xy + y^2$ equal?
8. What does $x^2 - 2xy + y^2$ equal?
9. If $x^2 + 2xy + y^2 = 49$,
and $x^2 + y^2 = 37$, what does $2xy$ equal?
10. If $x^2 + xy = 24$,
and $xy + y^2 = 12$, what does $x^2 + 2xy + y^2$ equal?
11. If $x^2 + 2xy + y^2 = 36$, what does $x + y$ equal?

12. If $x^2 + xy = 24$,
and $x + y = 6$, what does x equal?

SUGGESTION.—Divide the members of the first equation by those of the second.

13. If $xy + y^2 = 12$,
and $x + y = 6$, what is the value of y?

14. If $x^2 + y^2 = 2xy$, what does $x^2 - 2xy + y^2$ equal?

15. If $x^2 - 2xy + y^2 = 0$, what does $x - y$ equal?

16. If $x^2 + 2xy + y^2 = 36$,
and $4xy = 32$, what does $x^2 - 2xy + y^2$ equal?

17. What must be subtracted from $x^2 + 6xy + 9y^2$, the square of $x + 3y$, to get $x^2 - 6xy + 9y^2$, the square of $x - 3y$?

18. If $4x^2 + 4xy + y^2 = 49$,
and $xy = 6$, what does $4x^2 - 4xy + y^2$ equal?

19. If $x^2 + 6xy + 9y^2 = 64$,
and $xy = 5$, what will the square of $x - 3y$ equal?

157. The following equations are intended to be solved by first finding the values of $x + y$ and $x - y$. This will generally be done by finding the value of $x^2 + 2xy + y^2$ or $x^2 - 2xy + y^2$, and then taking the square root.

The student must use his ingenuity in combining the equations by addition, subtraction, or division, so as to produce the desired result by the simplest process.

Solve the following pairs of equations:

WRITTEN EXAMPLES.

1. $\begin{cases} x + y = 7 \\ x^2 - 2xy + y^2 = 25 \end{cases}$.

3. $\begin{cases} x^2 + 2xy + y^2 = 121 \\ 4xy = 112 \end{cases}$.

2. $\begin{cases} x^2 - y^2 = 5 \\ x - y = 1 \end{cases}$.

4. $\begin{cases} x + y = 8 \\ xy = 15 \end{cases}$.

5. $\left\{\begin{array}{r} x^2+y^2=10 \\ xy=\ \,3 \end{array}\right\}.$

6. $\left\{\begin{array}{r} x^2+xy+y^2=39 \\ x-y=\ \,3 \end{array}\right\}.$

7. $\left\{\begin{array}{r} x^2+xy=21 \\ xy+y^2=28 \end{array}\right\}.$

8. $\left\{\begin{array}{r} x-y=\ \,7 \\ xy=18 \end{array}\right\}.$

9. $\left\{\begin{array}{r} x^2-2xy+y^2=\ \,4 \\ x^2-y^2=24 \end{array}\right\}.$

10. $\left\{\begin{array}{r} x^2+4xy+4y^2=36 \\ x-2y=\ \,2 \end{array}\right\}.$

11. $\left\{\begin{array}{r} 3x+y=15 \\ xy=12 \end{array}\right\}.$

12. $\left\{\begin{array}{r} \dfrac{x}{y}+\dfrac{y}{x}=2 \\ x+y=8 \end{array}\right\}.$

13. $\left\{\begin{array}{r} xy+x-y=17 \\ xy=15 \end{array}\right\}.$

14. $\left\{\begin{array}{r} x^3-y^3=56 \\ x-y=\ \,2 \end{array}\right\}.$

SECTION LXIX.

PROBLEMS PRODUCING PURE QUADRATIC EQUATIONS.

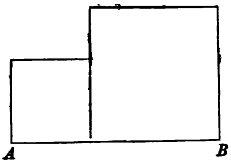
1. A MASON builds a wall at a certain number of dollars per yard, the number of yards being 6 more than the dollars per yard for building. He also builds another wall 6 yards long at the same price per yard, receiving $100 less for it than for the other. How much per yard did the walls cost?

SUGGESTION.—Let $x =$ the number of dollars per yard. Then $x + 6 =$ number of yards in first wall. Find expression for cost of second wall. Then the difference of cost will equal $100.

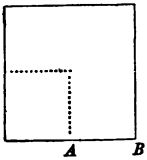
2. A person bought a field, paying 3 times as much per acre as the field contained acres. He then sold $\frac{1}{4}$ of the field for $\frac{2}{3}$ of the price per acre he paid for it. The remaining part thus cost him $640. How many acres did the field contain, and how much per acre did he pay for it?

SUGGESTION.—Let $x =$ number of acres in the field. Find expressions for price per acre, cost of whole, number of acres sold, price per acre of part sold, and whole price received for part sold.

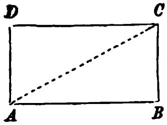
3. A man has two square fields adjoining each other; the two together contain 100 square rods, and the length of their two sides, from A to B, is 14 rods. What are the lengths of the sides of each?



4. A man has a square plot of ground, from one corner of which he cuts a square field, so as to leave him 95 square rods, and make the distance from A to B, 5 rods. What are the dimensions of the two squares?



5. A rectangular piece of ground contains 640 square yards, and its two adjacent sides, AB and DC, are together 52 yards long. What are its dimensions?



6. A man has material for 260 feet of fence, and wishes to enclose with it 3000 square feet in the form of a rectangle. What should be the sides?

7. A carpenter agrees to build a fence for \$48. But the proprietor decides to shorten the length 2 rods, and to pay \$2 more per rod, the fence thus costing him \$60. What was the length and what the cost per rod of the fence?

SUGGESTION.—Let x represent the length, and y the price per rod first agreed upon; then $x - 2$ and $y + 2$ represent the length and price actually decided upon.

8. A rectangular field contains 108 square rods, and the diagonal, AC, is 15 rods. What are the lengths of the sides?

SUGGESTION.—From the property of the right-angled triangle, ACB, the square of AC is equal to the sum of the squares of AB and BC.

9. A person purchased a piece of dry goods, paying for it as many cents a yard as there were yards. He then purchased another piece, containing 4 yards less than the first, and costing 4 cents a yard more. The two together cost \$1.84. What were the yards and price of each piece?

SECTION LXX.

ADFECTED QUADRATIC EQUATIONS.

158. QUADRATIC equations which contain both *first* and *second* powers of one unknown quantity are called **Adfected Quadratic Equations.**

As, $x^2 + 8x = 9$, or $x^2 + 5x = 3x + 4x^2 - 28$.

Such equations can often be reduced to the first degree by taking the square root of both members. Thus, if we have

$$x^2 + 4x + 4 = 49.$$

Taking the square root of both members, $x + 2 = \pm 7$.

Transposing, using first the + sign, and then the − sign, $x = -2 + 7$, or $-2 - 7$.

Collecting, we find $x = 5$, or -9.

Ex. Given $x^2 - 6x + 9 = 4$, to find the values of x.

SOLUTION. Taking the square root of both members,

$$x - 3 = \pm 2.$$

Transposing, $x = 3 + 2$, or $3 - 2$.

Collecting, we find $x = 5$, or 1.

It is usual to transpose and collect at one operation, and it is important to be able to do it readily and accurately.

Give the two values of x in each of the following equations:

MENTAL EXERCISE.

1. $x - 4 = \pm 2$.
2. $x + 3 = \pm 5$.
3. $x + 8 = \pm 3$.
4. $x - 9 = \pm 5$.
5. $x - 10 = \pm 1$.
6. $x + 4 = \pm 10$.
7. $x + 5 = \pm 3$.
8. $x - 8 = \pm 1$.
9. $x - 1 = \pm 8$.
10. $x + 3 = \pm 2$.

159. To find whether we can take the square root of the first member of such an equation as

$$x^2 - 6x + 9 = 36,$$

we proceed as in Art. 159. Divide twice the square root of the first term into the second, and the square of the result should be the third term.

MENTAL EXERCISE.

Tell which of the following equations can be reduced by taking the square root of the members, and perform the operation when possible:

1. $x^2 - 2x + 1 = 25.$
2. $x^2 - 8x + 16 = 49.$
3. $x^2 - 7x + \frac{49}{4} = \frac{81}{4}.$
4. $x^2 - 9x + \frac{81}{4} = \frac{121}{4}.$
5. $x^2 + 4x + 16 = 36.$
6. $x^2 + x + \frac{1}{4} = \frac{9}{4}.$
7. $x^2 - 9x + 9 = 49.$
8. $x^2 + 8x + 16 = 81.$

9. In the equation $x^2 - 6x + 4 = 11$, what ought the third term to be in order that the first member should be a perfect square?

10. What number must be added to each member of the equation $x^2 - 8x + 12 = 21$ to make the first member a perfect square? and what would the equation become if it were added?

11. In the equation $x^2 + 4x = 5$, what third term is wanted to make the first member a perfect square?

12. In the equation $x^2 - 8x = 20$, what third term is wanted? and what will the equation become after it is added to *both* members?

13. If a quantity be added to one member of an equation, why must it be added to the other also?

14. In the equation $x^2 + 12x = 13$, what third term is required to make the first member a perfect square? and

what will the equation become after it is added to both members?

Add to both members of the following equations the number which will make the first member of each a perfect square:

MENTAL EXERCISE.

1. $x^2 + 6x = 40$.
2. $9x^2 - 6x = 8$.
3. $x^2 - 12x = -11$.
4. $x^2 + 4x = 12$.
5. $x^2 + 3x = \frac{55}{4}$.
6. $x^2 + x = \frac{15}{4}$.
7. $25x^2 + 20x = 12$.
8. $16x^2 + 24x = 27$.

SECTION LXXI.

SOLUTION OF ADFECTED QUADRATIC EQUATIONS.

160. The foregoing operation of making the first member of an adfected quadratic equation a perfect square is called **completing the square.**

Ex. Given $x^2 + 6x = 40$; to find the value of x.

Solution. Completing the square by adding 9 to each member, we have

$$x^2 + 6x + 9 = 49.$$

Extracting the square root of both members,

$$x + 3 = \pm 7.$$

Transposing and collecting, we find

$$x = 4 \text{ or } -10.$$

Proof. Substituting these values in the first equation, we have

$$16 + 24 = 40,$$

$$\text{or,} \quad 100 - 60 = 40.$$

***Rule.**—Add to both members the number which will complete the square of the first member; extract the square root of both members, and transpose, collect, and reduce.*

Solve the following equations:

WRITTEN EXAMPLES.

1. $x^2 + 2x = 8$.
2. $x^2 + 2x = \frac{5}{4}$.
3. $x^2 - 10x = 24$.
4. $x^2 - 10x = -16$.
5. $x^2 + 6x = 7$.
6. $4x^2 - 12x = -8$.
7. $9x^2 + 6x = 15$.
8. $16x^2 - 24x = 40$.
9. $x^2 + x = 12$.
10. $x^2 + 3x = 18$.

161. If x^2 has a coefficient, as in the equation

$$3x^2 - 4x = 7,$$

the first term may be made a perfect square either by dividing both members by 3, obtaining

$$x^2 - \tfrac{4}{3}x = \tfrac{7}{3},$$

or, by multiplying by 3, obtaining

$$9x^2 - 12x = 21.$$

Both of these equations can be solved by the preceding method.

NOTE.—An adfected quadratic equation is in its *general form* when the terms containing x^2 and x, and those not containing x, are collected and arranged as in the above examples.

162. From the preceding operations we derive, for solving Adfected Quadratic Equations, the following

***Rule.**—I. Arrange the terms in the general form, and make the first term a square by multiplying or dividing each term of the equation by the coefficient of the first term.*

II. Complete the square by dividing twice the square root of the first term into the second, and adding the square of the result to both members of the equation.

III. Take the square root of both members, giving the root of the second member the double sign ±, and solve as in simple equations.

IV. Prove the result by substituting each value in the original equation.

NOTE.—When necessary, equations must be cleared of fractions, and indicated operations performed.

Solve the following equations:

WRITTEN EXAMPLES.

1. $x^2 - 6x = 72.$
2. $x^2 + 12x = 28.$
3. $2x^2 + 20x = 48.$
4. $x^2 + 5x = 14.$
5. $x^2 - 16x = 80.$
6. $x^2 + 4x = -3.$
7. $16x^2 + 32x = 33.$
8. $4x^2 + 4x = 3.$
9. $x^2 - x = 30.$
10. $x^2 + x = 12.$
11. $(x - 4)(x + 3) = -6.$
12. $x - 12 = -\frac{32}{x}.$
13. $\frac{x^2 - x}{2} - 2 = \frac{x^2}{4}.$
14. $3x^2 - 16x = 12.$
15. $2x - 6 = \frac{15 + x^2}{2x - 6}.$
16. $x - 5 = \frac{21}{x - 1}.$
17. $\frac{2x}{3} - \frac{2}{3} = \frac{x}{2} + 1 - \frac{4}{x}.$
18. $6x^2 - 13x = -6.$
19. $x - 3 = \frac{4 + x}{2x}.$
20. $\frac{x + 3}{8} = \frac{x}{x - 3}.$

163. If we take an equation like $3x^2 - 8x = 3,$

and make the first term a square by multiplying each term by 3, $9x^2 - 24x = 9,$

and complete the square by adding 16 to both members, we have $9x^2 - 24x + 16 = 25.$

It will be observed that the quantity, 16, added to each member to complete the square, is the square of 4, which is half the original coefficient of x. Likewise in every case, if we multiply each term by the coefficient of x^2, the quantity to be added to complete the square is the square of half the original coefficient of x.

For completing the square we have the following convenient

***Rule.*—Multiply each term by the coefficient of x^2, and add to each member the square of half the original coefficient of x.**

NOTE.—Where the coefficient of x^2 is 1, the multiplication may be omitted.

Ex. Given $5x^2 - 6x = 8$, to find x.

SOLUTION. Multiplying by 5, we have $25x^2 - 30x = 40$.

Adding 9, which is the square of $\frac{1}{2}$ of 6,

$$25x^2 - 30x + 9 = 49.$$

We will find, on applying the test, that the first member is now a perfect square.

Ex. Given $6x^2 + 5x = \frac{7}{3}$, to find x.

Multiplying by 6, $36x^2 + 30x = 14$.

Completing the square by adding the square of $\frac{5}{2}$, or $\frac{25}{4}$, $36x^2 + 30x + \frac{25}{4} = \frac{81}{4}$.

Taking the square root, $6x + \frac{5}{2} = \pm\frac{9}{2}$.

Transposing, $6x = -\frac{5}{2} + \frac{9}{2}$, or $-\frac{5}{2} - \frac{9}{2}$.

Collecting, $6x = 2$, or -7.

Dividing by 6, $x = \frac{1}{3}$, or $-\frac{7}{6}$.

Proof. Substituting these values in the first equation,

$$\frac{6}{9} + \frac{5}{3} = \frac{7}{3};$$

$$\frac{49}{6} - \frac{35}{6} = \frac{7}{3}.$$

Solve the following equations by this method:

WRITTEN EXAMPLES.

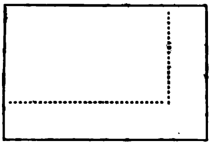
1. $3x^2 - 8x = -5$.
2. $2x^2 + 7x = 39$.
3. $x^2 + 4x = 45$.
4. $8x^2 - x = \frac{1}{4}$.
5. $x(5x - 8) = 4$.
6. $7x^2 - 4x = 51$.
7. $3x^2 + 3x = 6$.
8. $x(x - 6) = 7$.
9. $\frac{1}{x} - \frac{1}{x+3} = \frac{1}{6}$.
10. $\frac{x}{x-2} + \frac{5x+1}{x+3} = 5$.
11. $4x - \frac{14 - x}{x+1} = 14$.
12. $\frac{x^2 - 1}{4} = 1 - x$.
13. $x^2 - 3x = \frac{x^2 + 4}{5}$.
14. $5x^2 - 8x = -\frac{7}{5}$.
15. $6x^2 = 5(5 - x)$.

SECTION LXXII.

PROBLEMS PRODUCING QUADRATIC EQUATIONS.

1. A **RECTANGULAR** piece of ground is 4 rods longer than it is wide, and contains 60 square rods. What are its length and breadth?

2. A gentleman has a garden 50 yards long and 30 yards wide. He wishes to make a drive around two of the adjoining sides which will occupy $\frac{1}{4}$ of the ground. How wide must the drive be?



3. A lady purchased for \$3.20 a piece of muslin, which cost as many cents per yard as 4 less than the number of yards in the piece. What was the number of yards, and what the price per yard?

4. A certain number of gentlemen agree to contribute equal parts of \$240; but two having withdrawn, the share of each was increased \$6. How many gentlemen were there at first?

5. At a New Year's party every person gave a present to every other person, and there were 72 presents given. How many persons were there?

SUGGESTION.—If there were x people, find number of presents each gave.

6. A boat sails up stream 24 miles, and returns at a rate 2 miles per hour faster, occupying 1 hour less than was used in going. What were the rates of sailing?

7. A man walks 36 miles one day, and returns the next, walking 1 mile an hour slower and occupying 3 hours more. At what rates did he walk?

8. The members of a class desired to exchange photographs, and found that it would take 56 photographs. How many were in the class?

9. A man can do a piece of work in a certain number of hours, and his son in 8 hours longer. Both together can complete the work in 3 hours. In how many hours could each do the work?

10. It will take B 4 days longer to do a piece of work than A. A works at it 3 days, and then B finishes it in 5 days. In what time could each have done it alone?

11. A person intended to distribute 72 cents among a party of children, but the number of children being increased by 2, each child received 3 cents less than it would have received. How many children were there at first?

12. A man had to ride 20 miles in 4 hours. He rode half the distance at one rate, $\frac{3}{5}$ of the remainder at 2 miles an hour faster, and the remaining distance at double his first rate, arriving at his destination just at the end of his time. At what rates did he travel?

SECTION LXXIII.

REDUCTION OF RADICALS.

164. The lowest ten square numbers are

1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

The lowest five cubic numbers are

1, 8, 27, 64, 125.

The student should be thoroughly familiar with these, should be able to give the lists forward or backward, or tell which number precedes or follows any given one, and should know the numbers which are their roots.

Give the powers and roots indicated in the following expressions:

MENTAL EXERCISE.

1. $(4\,ax)^2$.
2. $(9\,x^2y^3)^2$.
3. $(a^2b^4x^3)^4$.
4. $(c^4d^2x^2)^3$.
5. $\sqrt{4\,x^4y^2}$.
6. $\sqrt{9\,a^2x^4}$.
7. $\sqrt[3]{x^6y^3}$.
8. $\sqrt[5]{a^{10}b^{15}}$.
9. $\sqrt{4\,c^8d^2}$.
10. $\sqrt[3]{27\,m^9n^6}$.
11. $\sqrt{49\,a^6x^4}$.
12. $\sqrt[3]{8\,a^6x^3y^9}$.

165. The expression $3\sqrt{4}$ means, and is read, 3 *times the square root of* 4. The square root of 4 being 2, its value is 3 times 2, or 6.

The expression $5\sqrt[3]{27\,a^3}$ means, and is read, 5 *times the cube root of* $27\,a^3$. The cube root of $27\,a^3$ being $3\,a$, its value is 5 times $3\,a$, or $15\,a$.

In the same way find the values of the following expressions:

MENTAL EXERCISE.

1. $4\sqrt{9}$.
2. $7\sqrt{25}$.
3. $8\sqrt{4}$.
4. $5\sqrt[3]{8}$.
5. $7\sqrt{a^2}$.
6. $a\sqrt{b^2}$.
7. $4\sqrt{x^2}$.
8. $6ax\sqrt{4y^2}$.
9. $3b\sqrt[3]{b^6}$.
10. $5xy\sqrt{16x^2y^4}$.
11. $3a^2\sqrt{9a^2}$.
12. $8x\sqrt[3]{x^3y^6}$.

166. Multiplication is often indicated by a dot. Thus, $2 \,.\, 4 \,.\, 6$ means the same as $2 \times 4 \times 6$. The dot occupies less space, and may be used in all cases instead of the sign ×, except between *two* numbers, as $6 \,.\, 8$, where it may be mistaken for a decimal point.

The expression $\sqrt{4} \,.\, \sqrt{9}$ means, and is read, the square root of 4 times the square root of 9. This is evidently the same as 2 times 3, or 6.

The expression $3\sqrt{25} \,.\, \sqrt{4}$ means, and is read, 3 times the square root of 25 times the square root of 4. It is equal to $3 \,.\, 5 \,.\, 2$, or 30.

Find the values of the following expressions:

MENTAL EXERCISE.

1. $\sqrt{4} \,.\, \sqrt{49}$.
2. $2\sqrt{16} \,.\, \sqrt{9}$.
3. $4\sqrt{a^2} \,.\, \sqrt{x^4}$.
4. $2\sqrt[3]{8} \,.\, \sqrt[3]{b^6}$.
5. $5\sqrt{4a^4} \,.\, \sqrt{x^2}$.
6. $3\sqrt[3]{27c^6} \,.\, \sqrt[3]{x^3}$.

167. Ex. What is the square root of 7?

Solution. We know that no exact number multiplied by itself will give 7. We must therefore indicate the root by the radical sign; thus, $\sqrt{7}$, read, *the square root of* 7. So we indicate the square root of a by $\sqrt{a}$, and the cube root of $6ax$ by $\sqrt[3]{6ax}$.

Ex. What is the value of $3\sqrt{4} \cdot \sqrt{7}$?

Solution. The square root of 4 is 2. We cannot take the square root of 7. Therefore, $3\sqrt{4} \cdot \sqrt{7} = 3 \cdot 2 \cdot \sqrt{7} = 6\sqrt{7}$.

Ex. What is the simplest form for the expression $5\sqrt{4a^2} \cdot \sqrt{3x}$?

Solution. $5\sqrt{4a^2} \cdot \sqrt{3x} = 5 \cdot 2a \cdot \sqrt{3x} = 10a\sqrt{3x}$.

Find the simplest form for the following expressions:

MENTAL EXERCISE.

1. $3\sqrt{9} \cdot \sqrt{15}$.
2. $6\sqrt{49} \cdot \sqrt{3}$.
3. $8\sqrt{4x^2} \cdot \sqrt{b}$.
4. $ax\sqrt{x^2} \cdot \sqrt{y}$.
5. $3bc\sqrt{4c^2} \cdot \sqrt{x}$.
6. $4ay\sqrt{9x^2y^2} \cdot \sqrt{y}$.
7. $2cd\sqrt{25d^2} \cdot \sqrt{c}$.
8. $7b^3\sqrt{16b^4c^2} \cdot \sqrt{3xy}$.

168. The expression $\sqrt{4} \cdot \sqrt{9}$ is equal to $\sqrt{4 \times 9}$, or $\sqrt{36}$. For $\sqrt{4} \cdot \sqrt{9} = 2 \times 3 = 6$, and $\sqrt{36} = 6$.

So the expression $\sqrt[3]{a^3} \cdot \sqrt[3]{b^3}$ is equal to $\sqrt[3]{a^3 \times b^3} = \sqrt[3]{a^3b^3}$.

For $\sqrt[3]{a^3} \cdot \sqrt[3]{b^3} = a \cdot b = ab$, and $\sqrt[3]{a^3b^3} = ab$.

In the same way, $\sqrt{a^2} \cdot \sqrt{x^2} = \sqrt{a^2x^2}$; and therefore, $\sqrt{a^2x^2} = \sqrt{a^2} \cdot \sqrt{x^2}$, and $\sqrt{9y^2} = \sqrt{9} \cdot \sqrt{y^2}$.

Principle. *Any quantity under the radical sign may be factored, and the radical sign placed over each factor.*

Illustration. $\sqrt{4a} = \sqrt{4} \cdot \sqrt{a} = 2\sqrt{a}$;

$\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \cdot \sqrt{2} = 3\sqrt{2}$;

and $5\sqrt{12} = 5\sqrt{4 \times 3} = 5\sqrt{4} \cdot \sqrt{3} = 5 \cdot 2 \cdot \sqrt{3} = 10\sqrt{3}$.

Thus it is possible to take the indicated root of some factor of a quantity when we cannot take the root of the whole quantity.

That this may be done readily, it is important to be able to recognize square and cubic factors, and others of higher power.

If the student is perfectly familiar with the list of square and cubic numbers given at the beginning of this section, there will be no difficulty in finding the largest square or cubic factor in any ordinary given number. Thus, the largest square factor of 12 is 4, of 75 is 25, and of 18 is 9. The largest cubic factor of 16 is 8, and of 54 is 27.

We can take a root of any power of a letter into which the index of the root will go. Thus, we can take the square root a^2, a^4, a^6, etc.; the cube root of a^3, a^6, a^9, etc.; the 4th root of a^4, a^8, a^{12}, etc.

The largest square factor of a^3 is a^2, a being the other factor.

The largest square factor of x^7 is x^6.

The largest 4th-power factor of a^{11} is a^8, a^3 being the other factor.

In each of the following expressions give the two factors, of which one is the largest square factor:

MENTAL EXERCISE.

1. 20.
2. 50.
3. 64.
4. 300.
5. a^3.
6. a^2x^3.
7. a^5x^4.
8. ax^3y^2.
9. $9x^4y^5$.
10. $28a^5y^3$.
11. $45b^3c$.
12. $108a^3b^2x$.

NOTE.—The required factors of $\frac{3}{4}$ would be $\frac{1}{4}$ and 3; of $\frac{8}{9}$ would be $\frac{4}{9}$ and 2.

13. $\frac{5}{4}$.
14. $\frac{7}{9}$.
15. $\frac{12}{25}$.
16. $\frac{18}{49}$.
17. $\frac{3}{64}$.
18. $\frac{50}{36}$.

For each of the following expressions give two factors, of which one is the largest cubic factor:

MENTAL EXERCISE.

1. 16.
2. 108.
3. $8ax^3$.
4. $27c^4d^6$.
5. $250x^2y^4z^3$.
6. $48ab^5c^8$.

169. By factoring in this way, the radical part of an expression may often be made more simple.

Ex. Simplify the expression $\sqrt{18}$.

Solution. We observe that 18 has a square factor, 9; therefore, $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} . \sqrt{2} = 3\sqrt{2}$.

Ex. Simplify the expression $3\sqrt[3]{24a^4x^2}$.

Solution. We observe that $24a^4x^2$ has a cubic factor, $8a^3$; therefore, $3\sqrt[3]{24a^4x^2} = 3\sqrt[3]{8a^3 \times 3ax^2} = 3\sqrt[3]{8a^3} . \sqrt[3]{3ax^2} = 6a\sqrt[3]{3ax^2}$.

A radical that has all the possible roots taken of its factors is said to be in its **simplest form.**

***Rule.**—Find the greatest square, or cubic, factor, according as the indicated root is square or cube, and take the root of that factor, indicating the root of the other factor as before.*

Reduce the following radicals to their simplest forms:

MENTAL EXERCISE.

1. $\sqrt{8}$.
2. $\sqrt{18}$.
3. $2\sqrt{50}$.
4. $5\sqrt{28}$.
5. $3\sqrt{32}$.
6. $\sqrt{a^2x}$.
7. $\sqrt{x^2y^3}$.
8. $3a\sqrt{a^3x^2}$.
9. $5b\sqrt{b^2y}$.
10. $2x\sqrt{20x^2}$.
11. $\sqrt[3]{a^3x^2}$.
12. $4\sqrt[3]{8a^4}$.
13. $7ay\sqrt{75a^2x^5}$.
14. $8\sqrt[3]{27x^3y^4}$.
15. $3\sqrt{16xy^3}$.

16. $5\sqrt[3]{32}$.
17. $4\sqrt{72}$.
18. $2a\sqrt{8a^2b}$.
19. $3\sqrt{4a^2b^3c^4}$.
20. $7x\sqrt[3]{a^3b^4x^2}$.
21. $ax\sqrt{16a^3x^2}$.
22. $xy\sqrt{200ax^2}$.
23. $6\sqrt{54a^2x^3y^4}$.
24. $3\sqrt[3]{16ay^4}$.
25. $4\sqrt{28b^3c^3}$.
26. $\sqrt{\frac{3}{4}}$.
27. $\sqrt{\frac{8}{9}}$.
28. $2\sqrt{\frac{15}{16}}$.
29. $3\sqrt{\frac{50}{36}}$.
30. $5\sqrt{\frac{3}{x^2}}$.
31. $9\sqrt{\frac{4a}{9b^2}}$.
32. $bc\sqrt{\frac{a^2c}{b^2}}$.
33. $3\sqrt{\frac{16x^2}{25}}$.

170. In order that a radical may be in its simplest form there must be no fractional part under the radical sign.

If the fraction under the radical sign does not have its denominator a square, it can be always made so by multiplying both numerator and denominator by the denominator, or, sometimes, by a less quantity.

Thus, $\sqrt{\frac{1}{2}} = \sqrt{\frac{1 \times 2}{2 \times 2}} = \sqrt{\frac{2}{4}}$,

and $5\sqrt{\frac{3}{8}} = 5\sqrt{\frac{3 \times 2}{8 \times 2}} = 5\sqrt{\frac{6}{16}}$.

These may now be reduced to their simplest forms as before.

Thus, $\sqrt{\frac{2}{4}} = \sqrt{\frac{1}{4} \times 2} = \frac{1}{2}\sqrt{2}$,

and $5\sqrt{\frac{6}{16}} = 5\sqrt{\frac{1}{16} \times 6} = \frac{5}{4}\sqrt{6}$.

Reduce the following radicals to their simplest forms:

MENTAL EXERCISE.

1. $\sqrt{\frac{1}{7}}$.
2. $\sqrt{\frac{1}{5}}$.
3. $\sqrt{\frac{2}{3}}$.
4. $8\sqrt{\frac{1}{8}}$.
5. $\sqrt{\frac{8}{5}}$.
6. $\sqrt{\frac{5}{7}}$.
7. $5\sqrt{\frac{1}{2}}$.
8. $3\sqrt{\frac{1}{6}}$.
9. $7\sqrt{\frac{3}{5}}$.

SECTION LXXIV.

ADDITION AND SUBTRACTION OF RADICALS.

171. Ex. What is the sum of $5\sqrt{3}$ and $7\sqrt{3}$?

SOLUTION. $5\sqrt{3} + 7\sqrt{3} = 12\sqrt{3}$, just as $5x + 7x = 12x$.

Ex. Subtract $4\sqrt{7}$ from $6\sqrt{7}$.

SOLUTION. $6\sqrt{7} - 4\sqrt{7} = 2\sqrt{7}$, just as $6x - 4x = 2x$.

Perform the following indicated addition and subtraction:

1. $2\sqrt{5} + 4\sqrt{5}$.
2. $7\sqrt{3} - 4\sqrt{3}$.
3. $\sqrt[3]{10} + 8\sqrt[3]{10}$.
4. $5\sqrt{7} - \sqrt{7}$.
5. $6\sqrt{15} + 2\sqrt{15}$.
6. $2\sqrt[3]{ax} - \sqrt[3]{ax}$.

Expressions which have the radical parts exactly alike are called *similar radicals;* as, $4\sqrt{5a}$ and $9\sqrt{5a}$.

Radicals like $3\sqrt[3]{7}$ and $4\sqrt[3]{10}$, or $5\sqrt{3}$ and $5\sqrt[3]{3}$ are *dissimilar,* and cannot be collected into one term.

7. $3\sqrt{2} + 5\sqrt{5} + 4\sqrt{2} - 3\sqrt{5}$.
8. $4\sqrt[3]{10} + 3\sqrt{10} - 2\sqrt[3]{10} + 7\sqrt{10}$.
9. $12\sqrt{ax} - 5\sqrt{2x} - 8\sqrt{ax} - 2\sqrt{2x}$.
10. $7\sqrt{5x} + 4\sqrt{15} - \sqrt{5x}$.

172. Radicals that are not similar may sometimes be made so by reducing them to their simplest forms. Thus, $\sqrt{8}$ and $3\sqrt{2}$ are dissimilar radicals, but $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$. $2\sqrt{2}$ and $3\sqrt{2}$ are similar radicals.

For addition and subtraction of radicals we have the following general

Rule.—*Reduce the radicals to their simplest forms; collect* *the similar radicals, and indicate the addition or subtraction of dissimilar ones by the proper signs.*

Ex. Add $\sqrt{75}$, $2\sqrt{12}$, and $\sqrt{3}$.

OPERATION.

$$\begin{array}{rcrcr}
\sqrt{75} &=& \sqrt{25 \times 3} &=& 5\sqrt{3} \\
2\sqrt{12} &=& 2\sqrt{4 \times 3} &=& 4\sqrt{3} \\
\sqrt{3} & & &=& \sqrt{3} \\
\hline
 & & & & 10\sqrt{3}
\end{array}$$

Explanation. We first reduce the radicals to their simplest forms, and, as they are then all similar, we add the coefficients, obtaining for the sum $10\sqrt{3}$.

Ex. Collect $2\sqrt[3]{135} + 2\sqrt{112} + \sqrt{7} - \sqrt[3]{40}$.

OPERATION.

We first reduce the radicals separately to their simplest forms.

$$\begin{array}{rcrcrcr}
2\sqrt[3]{135} &=& 2\sqrt[3]{27 \times 5} &=& 2.3.\sqrt[3]{5} &=& 6\sqrt[3]{5}, \\
2\sqrt{112} &=& 2\sqrt{16 \times 7} &=& 2.4.\sqrt{7} &=& 8\sqrt{7}, \\
-\sqrt[3]{40} &=& -\sqrt[3]{8 \times 5} & & &=& -2\sqrt[3]{5}. \\
\sqrt{7} &=& & & & & \sqrt{7}. \\
\hline
 & & & & & & 4\sqrt[3]{5}+9\sqrt{7}
\end{array}$$

WRITTEN EXAMPLES.

1. Add $3\sqrt{54}$, $\sqrt{24}$, and $\sqrt{6}$.
2. Subtract $\sqrt{18}$ from $2\sqrt{72}$.
3. Collect $\sqrt{20} - 2\sqrt{20} + 5\sqrt{45}$.
4. Add $\sqrt[3]{54}$ and $\sqrt[3]{250}$.
5. Collect $2\sqrt{4a} + \sqrt{9a} - \sqrt{25a}$.
6. Subtract $a\sqrt{4x}$ from $5\sqrt{a^2x}$.
7. Collect $4\sqrt[3]{2a} + 3\sqrt[3]{16a} - \sqrt[3]{54a}$.
8. Add $\sqrt{250}$, $\sqrt{640}$, and $\sqrt{1000}$.

9. Collect $6\sqrt{\frac{1}{3}} + 2\sqrt{12} - 8\sqrt{\frac{3}{4}}$.

10. Subtract $10\sqrt{\frac{1}{2}}$ from $7\sqrt{8}$.

Collect the following expressions:

MENTAL EXERCISE.

1. $2\sqrt{\frac{1}{2}} + 3\sqrt{2}$.
2. $4\sqrt{3} + \sqrt{75}$.
3. $4\sqrt{3} - \sqrt{27}$.
4. $6\sqrt{\frac{1}{3}} + 4\sqrt{3}$.
5. $2a\sqrt{x} + 3\sqrt{a^2x}$.
6. $2\sqrt{5x^3} + 6x\sqrt{5x}$.

SECTION LXXV.

REDUCTION OF RADICALS (Continued).

173. Ex. 3 is the square root of what quantity?

SOLUTION. 3 is the square root of 9. Hence, $3 = \sqrt{9}$. Here 3 has been reduced to a radical of the second degree.

Ex. $4a$ is the cube root of what quantity?

SOLUTION. $4a$ is the cube root of the cube of $4a$; that is, of $64a^3$. Therefore, $4a = \sqrt[3]{64a^3}$. Here $4a$ has been reduced to a radical of the third degree.

***Rule.**—Raise the quantity to the power indicated by the degree of the radical, and indicate the corresponding root.*

MENTAL EXERCISE.

1. Reduce xy to a radical of the 2d degree.
2. Reduce $3ax$ to a radical of the 3d degree.
3. Reduce $7a^3y$ to a radical of the 2d degree.
4. Reduce ax^3 to a radical of the 4th degree.

Ex. $3\sqrt{2}$ is the square root of what quantity?

SOLUTION. 3 is the square root of 9; $\sqrt{2}$ is the square root of 2; the square root of 9 times the square root of 2 is equal to the square root of 18.

Therefore, $3\sqrt{2} = \sqrt{9} \,.\, \sqrt{2} = \sqrt{18}.$

This operation is called *placing the coefficient under the radical sign.*

***Rule.**—Raise the coefficient to the power indicated by the degree of the radical, and multiply it by the part under the radical sign.*

Thus, $3\sqrt{a} = \sqrt{9a}$, $a\sqrt[3]{x} = \sqrt[3]{a^3x}$, and $3\sqrt{15} = \sqrt{135}$.

In the following expressions place the coefficients under the radical sign:

MENTAL EXERCISE.

1. $3\sqrt{7}$.
2. $4\sqrt{2}$.
3. $5a\sqrt{x}$.
4. $2a\sqrt[3]{y^2}$.
5. $6y\sqrt{x}$.
6. $5\sqrt{2}$.

SECTION LXXVI.

FRACTIONAL EXPONENTS.

174. THE square root of a^6 is a^3, of x^{10} is x^5. To find the square root of the power of any quantity we divide the exponent of that quantity by 2.

If the actual square root may be found by dividing the exponent by 2, the square root may be indicated by indicating the division of the exponent by 2. Thus, $a^{\frac{6}{2}}$ (read *a, six halves*) indicates the square root of a^6, while a^3 is the root actually taken. So $a^{\frac{6}{3}}$ (read *a, six thirds*) is the indicated cube root of a^6. By actual division the actual root, a^2, may be found.

Therefore, $a^{\frac{6}{2}}$ indicates the same operation as $\sqrt[2]{a^6}$,
and $a^{\frac{6}{3}}$ indicates the same operation as $\sqrt[3]{a^6}$.

Hence, $a^{\frac{6}{2}} = \sqrt{a^6}$, $a^{\frac{6}{3}} = \sqrt[3]{a^6}$, and $a^{\frac{8}{2}}b^{\frac{10}{2}}$ (read *a, eight halves; b, ten halves*) $= \sqrt{a^8b^{10}}$.

What roots are indicated in the following expressions? and what are the actual roots?—

MENTAL EXERCISE.

1. $a^{\frac{8}{2}}$.
2. $b^{\frac{9}{3}}$.
3. $a^{\frac{4}{2}}b^{\frac{6}{2}}$.
4. $4a^{\frac{10}{2}}$.
5. $x^{\frac{12}{3}}y^{\frac{6}{3}}$.
6. $a^{\frac{4}{2}}b^{\frac{10}{2}}c^{\frac{6}{2}}$.

175. In this way we may indicate roots which cannot be exactly found. Thus, since to get the square root we divide the exponent by 2, the square root of a^3 may be indicated by $a^{\frac{3}{2}}$ (read *a, three halves*). So $b^{\frac{4}{3}}$ indicates the third, or cube, root of b^4. Therefore $a^{\frac{3}{2}} = \sqrt{a^3}$ and $b^{\frac{4}{3}} = \sqrt[3]{b^4}$. To indicate the square root of a, $(= a^1)$, we divide the exponent 1 by 2, and have $a^{\frac{1}{2}}$. So $b^{\frac{1}{3}}$ indicates the cube root of b, $(= b^1)$.

Principle.—*When a fractional exponent is used the denominator indicates root, and the numerator indicates power.*

NOTE.—$2a^{\frac{2}{3}} = 2\sqrt[3]{a^2}$ and $(2a)^{\frac{2}{3}} = \sqrt[3]{2a^2}$. When no parenthesis is used the exponent belongs to only one quantity.

MENTAL EXERCISE.

What roots are indicated in the following expressions?—

1. $a^{\frac{1}{2}}$.
2. $b^{\frac{3}{2}}$.
3. $a^{\frac{2}{3}}b^{\frac{5}{3}}$.
4. $7a^{\frac{1}{3}}$.
5. $5a^{\frac{1}{2}}b^{\frac{3}{2}}$.
6. $4x^{\frac{7}{3}}y^{\frac{5}{3}}$.
7. $5^{\frac{1}{3}}$.
8. $(2a^2x^3)^{\frac{1}{3}}$.
9. $(4xy^3)^{\frac{1}{2}}$.

What exponents will indicate the following roots?—

10. $\sqrt{y}$.
11. $3\sqrt[3]{x}$.
12. $\sqrt[4]{a^2}$.
13. $2\sqrt{x^5}$.
14. $\sqrt{2}$.
15. $6\sqrt[3]{x^2y^4}$.

SECTION LXXVII.

REDUCTION OF RADICALS TO THE SAME DEGREE.

176. Ex. Reduce the exponents of $a^{\frac{1}{2}}$ and $b^{\frac{1}{3}}$ to equivalent ones having the same denominator.

OPERATION.

$\frac{1}{2} = \frac{3}{6}$, $\frac{1}{3} = \frac{2}{6}$. Hence $a^{\frac{1}{2}} = a^{\frac{3}{6}}$ and $b^{\frac{1}{3}} = b^{\frac{2}{6}}$.
So $a^{\frac{1}{4}}$ and $b^{\frac{1}{2}}$ equal $a^{\frac{1}{4}}$ and $b^{\frac{2}{4}}$.

Reduce the exponents in the following sets of quantities to the same denominator:

MENTAL EXERCISE.

1. $x^{\frac{2}{3}}$ and $y^{\frac{1}{2}}$.
2. $x^{\frac{1}{2}}$ and $x^{\frac{1}{4}}$.
3. $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$, and $a^{\frac{1}{6}}$.
4. $a^{\frac{1}{3}}$ and $b^{\frac{1}{4}}$.
5. $c^{\frac{1}{2}}$ and $d^{\frac{1}{6}}$.
6. $a^{\frac{3}{2}}$ and $b^{\frac{1}{4}}$.

Since the denominator of the exponent indicates the degree of the root, if two quantities have exponents with the same denominator, they are radicals of the same degree. Hence the

***Rule.**—Express the radicals with fractional exponents; reduce the exponents to the same denominator, and, if necessary, express them with the radical sign.*

ILLUSTRATION. $\sqrt{a}$ and $\sqrt[3]{a^2} = a^{\frac{1}{2}}$, and $a^{\frac{2}{3}} = a^{\frac{3}{6}}$, and $a^{\frac{4}{6}} = \sqrt[6]{a^3}$, and $\sqrt[6]{a^4}$.

Reduce the radicals in each of the following sets to the same degree:

WRITTEN EXAMPLES.

1. $\sqrt{5}$ and $\sqrt[3]{3}$.
2. $\sqrt{ax^3}$ and $\sqrt[3]{x^5}$.
3. $4\sqrt[4]{a^8}$ and $2\sqrt{a^5}$.
4. $\sqrt[3]{x^2}$, $\sqrt{ax}$, and $\sqrt[6]{xy}$.
5. $\sqrt[4]{a^3}$, $\sqrt[8]{ax}$, and $\sqrt[3]{ax^2}$.
6. $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[5]{5}$.

SECTION LXXVIII.

MULTIPLICATION AND DIVISION OF RADICALS.

177. WE have seen that $\sqrt{4} \times \sqrt{9} = \sqrt{36}$. Therefore, $\sqrt{36} \div \sqrt{9} = \sqrt{4}$. So $8\sqrt[3]{15} \div 4\sqrt[3]{5} = 2\sqrt[3]{3}$. But $\sqrt{3} \times \sqrt[3]{4}$ or $\sqrt{6} \div \sqrt[3]{2}$ cannot be expressed in any simpler form. If two radicals are of the same degree, their corresponding parts may be multiplied or divided; but if the degree is different, we can only indicate the operation. But we have found it possible to reduce radicals to the same degree. Hence, for multiplication and division of radicals we have the following

Rule.*—Reduce the radicals to the same degree, and multiply, or divide, the corresponding parts. When necessary, reduce the result to its simplest form.*

ILLUSTRATION. $\sqrt{3} \times \sqrt{2} = \sqrt{6}$, $3\sqrt{a} \times 4\sqrt{x^3} = 12\sqrt{ax^3}$ $= 12x\sqrt{ax}$, $\dfrac{8\sqrt{a^3}}{2\sqrt{a^2}} = 4\sqrt{a}$.

Perform the multiplication or division indicated in the following:

MENTAL EXERCISE.

1. $3\sqrt{2} \times \sqrt{5}$.
2. $4\sqrt{7} \times 2\sqrt{3}$.
3. $3\sqrt[3]{2a} \times \sqrt[3]{3ax}$.
4. $\dfrac{4\sqrt{15}}{2\sqrt{5}}$.
5. $\dfrac{12\sqrt[3]{21}}{6\sqrt[3]{3}}$.
6. $\dfrac{\sqrt{10xy^3}}{\sqrt{5xy^2}}$.
7. $5\sqrt{22} \times 8\sqrt{\frac{1}{2}}$.
8. $\dfrac{20\sqrt{42}}{10\sqrt{6}}$.
9. $\dfrac{6\sqrt[3]{100x^2}}{3\sqrt[3]{25x}}$.

WRITTEN EXAMPLES.

1. $\sqrt{10} \times \sqrt{6}$.
2. $4\sqrt{14} \times \sqrt{2}$.
3. $\sqrt[3]{3} \times \sqrt{2}$.
4. $\sqrt{3a} \times \sqrt{6ax}$.
5. $\sqrt{ax} \times \sqrt{xy}$.
6. $\sqrt[3]{x^2y} \times \sqrt{ay}$.
7. $3\sqrt{\frac{1}{2}} \times 5\sqrt{6}$.
8. $\dfrac{15\sqrt{3ac}}{5\sqrt[3]{6a}}$.
9. $\sqrt[3]{\frac{1}{3}} \times \sqrt[3]{9}$.
10. $\dfrac{\sqrt{5}}{3\sqrt[3]{10}}$.
11. $\dfrac{8\sqrt[3]{a^2x^2}}{2\sqrt{x}}$.
12. $\dfrac{\sqrt{2a}}{\sqrt{\frac{1}{2}}}$.

SECTION LXXIX.

FRACTIONAL EXPONENTS.

178. THE power of any quantity is the result obtained by taking that quantity a given number of times as a factor. The exponent shows how many times the quantity is to be taken. Thus, x^5 is the 5th power of x, and is the product of 5 x's.

QUESTION. What power of x is indicated by $x^{\frac{1}{2}}$?

SOLUTION. An arithmetical fraction indicates a certain number of equal parts. Thus, $\frac{1}{2}$ indicates 1 of 2 equal parts; $\frac{3}{4}$ means 3 of 4 equal parts, and so on.

If we compare the meanings of $5x$ and x^5, we find that $5x$ means the *sum* of 5 x's, and that x^5 means the *product* of 5 x's.

If we compare $\frac{1}{2}x$ and $x^{\frac{1}{2}}$, we find the same relation. $\frac{1}{2}x$ means 1 of 2 parts whose *sum* is x, and $x^{\frac{1}{2}}$ means 1 of 2 parts whose *product* is x.

So $\frac{2}{3}x$ means the *sum* of 2 of the 3 equal parts whose *sum* is x, and $x^{\frac{2}{3}}$ means the *product* of 2 of the 3 equal parts whose *product* is x.

Fractional Exponents, therefore, have the same relation to entire exponents that fractional coefficients have to entire coefficients.

Since $x^{\frac{1}{2}}$ denotes 1 of 2 equal factors of x, it means the square root of x, and we have, as before, $x^{\frac{1}{2}} = \sqrt{x}$.

So $x^{\frac{1}{3}}$, which denotes 1 of *three* equal factors of x, means the cube root of x, and, as before, $x^{\frac{1}{3}} = \sqrt[3]{x}$.

The product of two equal factors is the square of one of them. Hence, $x^{\frac{2}{3}}$, being the product of *two* of the *three* equal factors of x, is the square of $x^{\frac{1}{3}}$, or the square of the cube root of x. Therefore, $x^{\frac{2}{3}} = (\sqrt[3]{x})^2 = \sqrt[3]{x^2}$.

SECTION LXXX.

NEGATIVE EXPONENTS.

MENTAL EXERCISE.

	1.	2.	3.	4.
	$5x$	$8x$	$9x$	$5x$
Subtract	$2x$	$4x$	$3x$	x

	5.	6.	7.	8.
Divide	$\frac{x^5}{x^2}$	$\frac{x^8}{x^4}$	$\frac{x^9}{x^3}$	$\frac{x^5}{x}$

179. In the *subtraction* we subtracted the *coefficients* because they showed how many times the letters were taken as added quantities, and the difference of the coefficients showed how many times the quantity was left.

In the *division* we subtracted the *exponents* because they showed how many times the letters were taken as multiplied quantities or factors, and the difference of the exponents showed how many factors were left after the division.

Let us now subtract

$$\begin{array}{ll} \text{Min.} & 2y \\ \text{Sub.} & \underline{5y} \\ \text{Rem.} & -3y \end{array}$$

Here the minuend is too small by $3y$ for $5y$ to be subtracted from it; we indicate this by writing $-3y$ for the result.

Now let us divide $\frac{y^2}{y^5}$

Here the dividend is too small by 3 y's as factors for y^5 to be divided into it; we indicate this absence of 3 factors by writing for the result y^{-3} (read *y, minus* 3, or, *minus* 3 power of *y*).

In the expression $-3y$, the coefficient -3 shows how many y's are lacking; and in the expression y^{-3}, the exponent -3 shows how many y's as factors are lacking.

180. Thus we see that positive and negative exponents have the same relation to each other as positive and negative coefficients.

These discussions also show us that all exponents indicate how many times the quantity is taken as a factor, and that therefore all quantities with exponents are powers.

For division of powers we have the following general

Rule.—*In all cases of division of powers of the same quantity subtract the exponent of the divisor from the exponent of the dividend to find the exponent of the quotient.*

MENTAL EXERCISE.

Explain the following:

1. $\frac{x^5}{x^2} = x^3$.
2. $\frac{x^2}{x^5} = x^{-3}$.
3. $\frac{a^8}{a^4} = a^4$.
4. $\frac{a^4}{a^8} = a^{-4}$.
5. $\frac{5xy^{-3}}{x^3y} = 5x^{-2}y^{-4}$.
6. $\frac{5x^3y}{xy^{-3}} = 5x^2y^4$.

Give similar results for the following indicated divisions:

1. $\frac{x^5}{x^2}$.
2. $\frac{x^4}{x^6}$.
3. $\frac{4x^3}{2x^6}$.
4. $\frac{6x^2y}{3xy}$.
5. $\frac{10ax^2y}{2ax^4}$.
6. $\frac{12x^2y}{3x^3y^4}$.
7. $\frac{8ax^4y}{8axy^5}$.
8. $\frac{x^3yz^4}{x^5y^2z}$.
9. $\frac{15ac^3x}{5a^2cx^4}$.
10. $\frac{xy^3}{x^4y}$.
11. $\frac{a^3}{a^{-2}}$.
12. $\frac{a^{-2}}{a^{-1}}$.
13. $\frac{x^{-4}}{x^2}$.
14. $\frac{a^{-2}x^3}{a^3x^{-1}}$.
15. $\frac{x^4y^{-2}}{x^2y^{-4}}$.

181. We will now indicate the division of a^3 by a series of ascending powers of a:

Obtaining $a^3, \frac{a^3}{a}, \frac{a^3}{a^2}, \frac{a^3}{a^3}, \frac{a^3}{a^4}, \frac{a^3}{a^5}, \frac{a^3}{a^6}$.

Performing the division by subtracting exponents, $a^3, a^2, a^1, a^0, a^{-1}, a^{-2}, a^{-3}$.

Reducing the same fractions to their lowest terms, $a^3, a^2, a, 1, \frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}$.

The corresponding terms of these three series are equal to each other; hence,

$$a^0 = 1.$$

$$a^{-1} = \frac{1}{a}.$$

$$a^{-2} = \frac{1}{a^2}.$$

$$a^{-3} = \frac{1}{a^3}.$$

Since a may stand for any quantity, the 0 *power* (*read zero power*) *of any quantity is* 1. *A negative power of any quantity is equal to* 1 *divided by the same positive power of the quantity.*

Thus, $x^{-2}=\frac{1}{x^2},\quad a^{-5}=\frac{1}{a^5},\quad y^{-1}=\frac{1}{y},$

and $4x^{-2}=\frac{4}{x^2},\quad ay^{-3}=\frac{a}{y^3},\quad bx^{-1}=\frac{b}{x}.$

Since $a^0=1,\quad 3a^0=3,$

and $4a^0x^2=4x^2.$

Express the following without any negative exponents:

MENTAL EXERCISE.

1. $3x^{-2}$.
2. $4ay^{-4}$.
3. $2xy^{-2}$.
4. $8a^{-2}b^{-1}$.
5. $9x^{-4}y^2$.
6. $10a^3c^{-2}$.
7. $\frac{12cdy^{-3}}{x}$.
8. $\frac{5a^3cx^{-1}}{y^3}$.
9. $\frac{m^2n^{-3}}{c^3}$.

182. We have found that $a^{-2}=\frac{1}{a^2}$; hence, $\frac{1}{a^{-2}}=\frac{1}{\frac{1}{a^2}}$ $=1\times\frac{a^2}{1}=a^2$. That is, $\frac{1}{a^{-2}}=a^2$. So $\frac{2a}{bx^{-1}}=\frac{2ax}{b}$.

Rule.*—We may transfer any factor from either numerator or denominator of a fraction to the other term by changing the sign of the exponent of that factor.*

ILLUSTRATION. $\frac{3ax^{-2}}{4b^{-3}y}=\frac{3ab^3}{4x^2y}$, and $\frac{a^{-2}}{x^3}=\frac{1}{a^2x^3}$.

MENTAL EXERCISE.

Reduce the following to equivalent expressions having no negative exponents:

1. $\dfrac{5ay^2}{3b^{-2}x^3}$. 3. $\dfrac{6a^4b^{-3}}{a^{-2}b^2}$. 5. $\dfrac{xy^{-2}}{a^{-2}x^{-1}}$.

2. $\dfrac{3m^{-3}n}{4a^{-4}b^{-2}}$. 4. $\dfrac{a^{-2}bc^3x}{ax^{-3}y}$. 6. $\dfrac{a^{-3}m}{x^{-2}y^4}$.

Reduce the following expressions to the form of entire quantities:

1. $\dfrac{2a^2y}{b^{-3}x^{-2}}$. 3. $\dfrac{5ax}{b^2y}$. 5. $\dfrac{12ac^2y^3}{6b^{-1}x^{-3}}$.

2. $\dfrac{9mn^4}{3a^{-1}}$. 4. $\dfrac{m^2x^3}{m^{-1}x^{-2}}$. 6. $\dfrac{a^4dn}{b^{-1}m}$.

SECTION LXXXI.

OPERATIONS WITH NEGATIVE AND FRACTIONAL EXPONENTS.

183. ALL the operations of Addition, Subtraction, Multiplication, and Division may be performed with terms containing negative and fractional exponents by exactly the same rules as were used in Part I.

If some quantities are expressed with fractional exponents, and others with the radical sign, they must be reduced to the same kind of expressions,

NOTE.—In addition and subtraction be careful to unite only similar terms.

MENTAL EXERCISE.

$$\begin{array}{llll} & 1. & 2. & 3. \\ \text{Add} & 4a^{\frac{1}{2}}x^{-2} - 3a^2x^{-5} & a^{-2}b^3 - 3y^{-2} & x^{-2} \\ & \underline{3a^{\frac{1}{2}}x^{-2} + 2a^2x^{-5}} & \underline{4a^{-2}b^3 - 2y^{-2}} & \underline{x^{\frac{3}{2}}} \end{array}$$

$$\begin{array}{lll} & 4. & 5. \\ \text{Subtract} & 5x^{-2}y^2 - 5x^{-3} & 7ab \quad + 8a^3c^{-2} + x^{-4} \\ & \underline{2x^{-2}y^2 + 2x^3} & \underline{7ab^{-1} - 2a^3c^{-3} + x^{-4}} \end{array}$$

In multiplication of powers of the same quantity add the exponents. In division subtract the exponent of the divisor from the exponent of the dividend.

NOTE.—If the exponent in the result is 0, the quantity is equal to 1, and is usually omitted as a factor.

ILLUSTRATION. $a^{-2} \times a^{-3} = a^{-5}$. $a^{\frac{1}{2}} \times a^{\frac{3}{2}} = a^{\frac{4}{2}} = a^2$.

$a^2x^{-3} \times bx^3 = a^2b$. $\frac{a^3}{a^{-2}} = a^5$. $\frac{a^{\frac{5}{2}}}{a^{\frac{3}{2}}} = a^{\frac{2}{2}} = a^1 = a$.

Perform the multiplication and division indicated in the following:

MENTAL EXERCISE.

1. $a^2x^3 \times a^3x^{-1}$.
2. $3c^{-2}d \times 2c^3d^{-4}$.
3. $5x^{-1}y^{-2} \times 3a^2x^{-2}$.
4. $6a^3c^{-1}x^{-2} \times c^{-2}x$.
5. $ay^2 \times 5a^3x^{-2}$.
6. $a^{-3} \times a^3$.
7. $a^{\frac{1}{2}}x \times a^{\frac{3}{2}}$.
8. $4a^{-\frac{1}{2}} \times a^{\frac{3}{2}}$.
9. $5x^{\frac{1}{2}}y^2 \times x^{\frac{1}{2}}y^{-1}$.
10. $3a^2x^{-2} \times a^{-1}x^2$.
11. $a^{-\frac{3}{2}}x^{\frac{2}{3}} \times a^{\frac{5}{2}}x^{\frac{1}{3}}$.
12. $x^{\frac{1}{2}} \times x^{\frac{1}{3}}$.
13. $\frac{x}{x^{\frac{1}{2}}}$.
14. $\frac{a}{a^{\frac{1}{3}}}$.
15. $\frac{x^{\frac{2}{3}}y^{\frac{3}{2}}}{x^{\frac{1}{3}}y}$.
16. $\frac{x^{\frac{3}{2}}y^{\frac{1}{3}}}{x^{-\frac{1}{2}}y^{\frac{2}{3}}}$.
17. $\frac{10c^{\frac{3}{4}}d^{\frac{2}{3}}}{5c^{\frac{1}{4}}d^{\frac{1}{3}}}$.
18. $\frac{2ay^3}{a^{\frac{1}{2}}y^{\frac{5}{2}}}$.

19. $\dfrac{m^{-\frac{1}{3}}y^2}{m^{-\frac{1}{2}}y^{-4}}$. 20. $\dfrac{8a^2x^{\frac{2}{3}}y^{-\frac{1}{2}}}{4a^{\frac{1}{2}}x^{-\frac{1}{3}}y^{-\frac{1}{2}}}$.

WRITTEN EXAMPLES.

1. Add $4a^{\frac{1}{2}} + 3ax^{-2} + 5\sqrt[3]{y}$, $2\sqrt{a} - \dfrac{5a}{x^2} + 2y^{\frac{1}{3}}$, and $\sqrt{9}a + \dfrac{2x^{-2}}{a^{-1}} - 3\sqrt[3]{y}$.

2. Add $4\sqrt{ax^2} - \dfrac{3}{b} + 2a^3c^{\frac{1}{2}}$, $4x\sqrt{a} + 2b^{-1} + a^3\sqrt{c}$, and $5a^{\frac{1}{2}}x - \dfrac{4}{b} - \dfrac{5\sqrt{c}}{a^{-3}}$.

3. Subtract $2b^{-1}x^{\frac{1}{2}} - 8b^{\frac{1}{2}}y^{\frac{2}{3}} + 2x^2y^{-1}$ from $\dfrac{6\sqrt{x}}{b} - 10\sqrt[3]{y^2}\cdot\sqrt{b} + \dfrac{4x^2}{y}$.

4. Subtract $\sqrt{xy^2} + 4x^2y + 2xy^{\frac{1}{2}}$ from $3y\sqrt{x} + \dfrac{2x^2}{y^{-1}} - 4x\sqrt{y}$.

5. Multiply $a^{\frac{1}{2}} + b^{\frac{1}{2}}$ by $a^{\frac{1}{2}} + b^{\frac{1}{2}}$.

6. Multiply $x^{-2} + x^{-1} + 1$ by $x + x^2$.

7. Multiply $x^{\frac{3}{2}} + x + x^{\frac{1}{2}}$ by $x^{\frac{1}{2}} + 1$.

8. Multiply $a^2 + 3a + a^{-1}$ by $2a + a^{-1}$.

9. Multiply $x^{\frac{3}{4}} + 2x^{\frac{1}{2}} + x^{\frac{1}{4}}$ by $x^{\frac{1}{2}} + x^{\frac{1}{4}}$.

10. Divide $a^{\frac{3}{2}} + 3ab^{\frac{1}{2}} + 3a^{\frac{1}{2}}b + b^{\frac{3}{2}}$ by $a^{\frac{1}{2}} + b^{\frac{1}{2}}$.

11. Divide $a - b$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.

12. Divide $x^{-3} + 3x^{-2} + 4x^{-1} + 2$ by $1 + x$.

13. Divide $x^{\frac{3}{2}} + 2x^{\frac{4}{3}} + 2x^{\frac{7}{6}} + x$ by $x^{\frac{1}{2}} + x^{\frac{1}{3}}$.

14. Divide $x^4 + x^2 + 1 - 3x^{-2}$ by $x^2 + 2x + 3x^{-1}$.

SECTION LXXXII.

SQUARES OF RADICALS.—RADICAL EQUATIONS.

184. If we wish to indicate the root of a polynomial we prefix the radical sign, and draw a line over the whole quantity. Thus, the square root of $a^2 + 2ab + b^2$ would be indicated by $\sqrt{a^2 + 2ab + b^2}$. The line is called a **vinculum.** It has the same effect as a parenthesis, and may often be used in place of one. We take the root only of the part over which the vinculum extends. Thus,

$$\sqrt{9} + 7 = 3 + 7 = 10, \qquad \sqrt{9 + 7} = \sqrt{16} = 4.$$

The quantities under the radical sign thus far have been monomials, but the same principles apply to polynomial radicals. Thus,

$$\sqrt{a^2 - 2ab + b^2} = a - b, \qquad \sqrt[3]{(a + x)^4} = \sqrt[3]{(a + x)^3(a + x)}$$
$$= (a + x)\sqrt[3]{a + x}.$$

MENTAL EXERCISE.

Give the values of the following expressions, taking only the positive roots:

1. $\sqrt{3 + 6}$.
2. $\sqrt{9} + 4$.
3. $\sqrt{16 + 9}$.
4. $\sqrt{16} + 9$.
5. $\sqrt{16} - \sqrt{9}$.
6. $16 - \sqrt{9}$.

Extract the roots indicated in the following expressions:

7. $\sqrt{x^2 + 2xy + y^2}$.
8. $\sqrt{a^2(b^2 + 2bc + c^2)}$.

Collect:

9. $4\sqrt{a + b} + 2\sqrt{a + b}$.
10. $5\sqrt{x^2 - y^2} - 3\sqrt{x^2 - y^2}$.

Perform the following indicated multiplications:

11. $\sqrt{x} \cdot \sqrt{2x - 2}$.
12. $\sqrt{x + 4} \cdot \sqrt{3}$.
13. $\sqrt{a + b} \cdot \sqrt{a - b}$.
14. $\sqrt{x - y} \cdot \sqrt{xy}$.

Perform the following indicated divisions:

15. $\dfrac{\sqrt{a^3 - x^3}}{\sqrt{a - x}}$.
16. $\dfrac{\sqrt{a^3 + 2a^2b + ab^2}}{\sqrt{a}}$.

185. The square root of a quantity multiplied by itself gives that quantity. Thus, $\sqrt{a} \times \sqrt{a}$ gives a. The square of the square root of any quantity is equal to the quantity. Thus, $(\sqrt{a})^2$ is equal to a.

PRINCIPLE.—*Squaring a radical of the second degree removes the radical sign.*

Give the squares of the following quantities:

MENTAL EXERCISE.

1. $\sqrt{5}$.
2. $\sqrt{14}$.
3. $2\sqrt{x}$.
4. $\sqrt{x + 4}$.
5. $3\sqrt{5x - 2}$.
6. $\dfrac{\sqrt{x+1}}{5}$.
7. $\sqrt{8x - \sqrt{x}}$.
8. $\sqrt{4x - x^2}$.
9. $4\sqrt{\dfrac{x - 2}{8}}$.

186. An expression like $4 + \sqrt{x}$, being the sum of two quantities, is best squared by use of Theorem I. The square would be the square of the first term, or 16, plus twice the product of the first and second, or twice $4\sqrt{x}$, which is $8\sqrt{x}$, plus the square of the second, or the square of $\sqrt{x}$, which is x. The result is $16 + 8\sqrt{x} + x$.

Ex. What is the square of $\sqrt{x} - \sqrt{x - 2}$?

SOLUTION. This is the difference of the two quantities, $\sqrt{x}$ and $\sqrt{x - 2}$. Its square is best found by Theorem II. It will be the square of $\sqrt{x}$, or x, minus twice the product of $\sqrt{x}$ and $\sqrt{x-2}$, or $2\sqrt{x^2-2x}$ plus the square of $\sqrt{x-2}$, or $x-2$. The result is $x+2\sqrt{x^2-2x}+x-2$.

MENTAL AND WRITTEN EXERCISE.

Find the squares of the following by the use of Theorem I. or Theorem II.:

1. $\sqrt{x}-2$.
2. $4-\sqrt{3x}$.
3. $2-3\sqrt{2x}$.
4. $\sqrt{x}-\sqrt{x+2}$.
5. $x+\sqrt{x-4}$.
6. $3x-\sqrt{9x^2-2x}$.

It will be observed that in all these cases the middle term of the result contains a radical expression.

Square both members of the following equations:

7. $\sqrt{x}=5$.
8. $\sqrt{x+1}=3$.
9. $\sqrt{x}=\sqrt{2x-8}$.
10. $\sqrt{4x}=\sqrt{x}+2$.

187. In squaring the members of these equations it will be observed that where the radical expression stands by itself the radical sign disappears, but that where it is connected with another quantity it is still found in the middle term of the square. Equations must be freed from radicals before they can be solved, and therefore it is necessary to place the radical by itself on one side of the equation before squaring the members.

If there are two or more radical expressions, as in the equation,

$$\sqrt{3x-1}-\sqrt{x-1}=2,$$

we transpose one to the other member, so as to leave the other by itself, obtaining

$$\sqrt{3x-1}=\sqrt{x-1}+2.$$

Squaring, $3x - 1 = x - 1 + 4\sqrt{x - 1} + 4.$

As a radical still remains, we transpose and collect the other terms, so as to leave the radical by itself, $2x - 2 = 4\sqrt{x - 1}.$

Dividing by 2, $x - 1 = 2\sqrt{x - 1}.$

Squaring, $x^2 - 2x + 1 = 4x - 4.$

Transposing and collecting, $x^2 - 6x = -5.$

Completing the square, $x - 6x + 9 = 4.$

Extracting the square root, $x - 3 = \pm 2.$

Transposing and collecting, $x = 5$ or $1.$

For solving equations containing radicals of the second degree we have the following

***Rule.**—I. Transpose the terms so that one radical may stand by itself on one side of the equation, and square both members.*

II. If any radical remains, repeat the operation till the resulting equation contains no radical.

III. Solve this equation by some of the preceding methods.

Solve the following equations:

WRITTEN EXAMPLES.

1. $\sqrt{x + 6} = 3.$
2. $\sqrt{8x + 4} = 3\sqrt{x}.$
3. $\sqrt{5 + x} = \sqrt{x} + 1.$
4. $3 + \sqrt{2x + 2} = 7.$
5. $1 + 2\sqrt{x - 1} = x.$
6. $\sqrt{x - 2} = \sqrt{x + 10} - 2.$
7. $\sqrt{x + 5 + \sqrt{x + 1}} = \sqrt{x + 8}.$
8. $x - \sqrt{2x + 1} = 1.$
9. $x - 1 = \sqrt{10 - 2x}.$
10. $\sqrt{4x + 1} - \sqrt{3} = \sqrt{12}.$
11. $x - 2 = \sqrt{19 - 2x}.$
12. $x - 6 = \sqrt{3(x - 6)}.$
13. $x - 1 = \sqrt{8x - 23}.$
14. $\sqrt{4 + \sqrt{4x + 5}} = 3.$

SECTION LXXXIII.

RATIO AND PROPORTION.

MENTAL EXERCISE.

1. 6 is how many times 3?
2. 8 is how many times 2?
3. 25 is how many times 5?
4. 4 is what part of 12?
5. 3 is what part of 15?
6. a is what part of $4a$?
7. 5 is what part of 8?
8. 6 is what part of 10?
9. 4 is what part of 6?
10. 8 is what part of 14?

188. In all of these examples we found the *relation* between the two quantities by dividing the first by the second. This relation is usually called **Ratio.**

In all the former questions we might have asked what was the *ratio* of one number to another. Thus, the ratio of 6 to 2 is 3; the ratio of 5 to 15 is $\frac{1}{3}$.

What is the ratio of—

11. 12 to 4?
12. 9 to 3?
13. 56 to 8?
14. 2 to 10?

Ratio is usually expressed by two dots, thus: $5:10$.

What is the value of the following ratios?—

15. $4:32$.
16. $\frac{1}{2}:2$.
17. $\frac{1}{2}:\frac{3}{2}$.
18. $\frac{2}{3}:\frac{1}{3}$.
19. $6:\frac{1}{2}$.
20. $2:12$.

Find to which ratio in the second column each ratio in the first column is equal:

1. $3:1$.
2. $4:6$.
3. $2:8$.
4. $5:\frac{1}{2}$.

5. $1:4$.
6. $2:\frac{2}{3}$.
7. $20:2$.
8. $6:9$.

189. When the equality of two or more ratios is indicated, it is usual to place four dots (: :) between the ratios, instead of the usual equal sign (=). Thus, the equality of the ratios of 5 : 10 and 1 : 2 is expressed, 5 : 10 : : 1 : 2. This is called a **proportion**, and is read, "5 is to 10 as 1 is to 2."

A **proportion** is an expressed equality of ratios.

Find to which ratio in the second column each ratio in the first is equal, and express the equality in the form of a proportion:

WRITTEN EXAMPLES.

1. 9 : 3.
2. 5 : 12.
3. 2 : 10.
4. $\frac{1}{2}$: 4.
5. 6 : $\frac{2}{3}$.
6. 3 : 15.
7. 10 : 24.
8. 18 : 2.
9. 1 : $\frac{1}{3}$.
10. 2 : 16.

190. Since the value of a ratio is found by dividing the first quantity by the second, the ratio may also be indicated by indicating the division. Thus, the ratio of 6 to 3 may be represented by $\frac{6}{3}$. Since the ratio of 4 : 12 is the same as 2 : 6, we must have 4 divided by 12 equal to 2 divided by 6, or, $\frac{4}{12} = \frac{2}{6}$.

Therefore the equation $\frac{4}{12} = \frac{2}{6}$ and the proportion 4 : 12 : : 2 : 6 are both statements of the fact that the relation of 4 to 12 is the same as the relation of 2 to 6.

PRINCIPLE.—*An equality of fractions may always be represented by a proportion, and a proportion by an equality of fractions.*

Change the fractional equations to proportions, and the proportions to fractional equations, in the following:

MENTAL EXERCISE.

1. $\frac{1}{3} = \frac{2}{6}$.
2. $\frac{a}{b} = \frac{c}{d}$.
3. $\frac{3}{4} = \frac{x}{y}$.
4. $\frac{x+y}{3} = \frac{x-2}{4}$.
5. $\frac{5y+1}{6} = \frac{3}{4}$.
6. $2 : 8 :: 3 : 12$.
7. $5 : 3 :: 15 : 9$.
8. $a : x :: b : y$.
9. $x + y : x :: 4 : 5$.
10. $x + 3 : x - 3 :: 8 : 2$.

191. The first and last terms of a proportion are called extremes, and the second and third terms are called means.

If the proportion $a : b :: c : d$ be expressed as an equality of fractions, we will have $\frac{a}{b} = \frac{c}{d}$. If we clear this equation of its fractions by multiplying both terms by bd, it becomes

$$ad = bc.$$

We observe that a and d were the extremes of the proportion, and b and c are the two means; and since the product of a and d is equal to the product of b and c, we have the

PRINCIPLE.—*In every proportion, the product of the extremes is equal to the product of the means.*

ILLUSTRATION. If $x : y :: 5 : 7$, then $7x = 5y$.

If $x + 4 : x - 4 :: 9 : 6$, then $6x + 24 = 9x - 36$.

Change the following proportions to equations, by placing the product of the extremes equal to the product of the means:

MENTAL EXERCISE.

1. $x : x - 3 :: 5 : 3$.
2. $x : y :: 2 : 7$.
3. $x + 3 : x - 5 :: 3 : 1$.
4. $\frac{x}{2} : 3x - 1 :: 1 : 4$.
5. $x + y : x - y :: 3 : 2$.
6. $x + 5 : 7 - x :: 5 : 3$.

Transform the following proportions to equations, and solve the equations:

WRITTEN EXAMPLES.

1. $2x + 9 : x + 2 :: 3 : 1.$

2. $x - 2 : 6 :: 2x : 15.$

3. $x - 1 : x + 2 :: 8 : 4.$

4. $\left\{\begin{array}{c} x + y : 2x :: 5 : 2 \\ 5x - y = 2 \end{array}\right\}.$

5. $\left\{\begin{array}{c} 2y + x : 1 :: 4x + y : 3 \\ 4x + y = 42 \end{array}\right\}.$

6. $\left\{\begin{array}{c} x + 2y : 2x + y :: 5 : 6 \\ \dfrac{2x + 3y}{2} = 2x - 1 \end{array}\right\}.$

7. $\left\{\begin{array}{c} x + 4y : 4x - 2 :: 1 : 2 \\ 5 : 3x + 5y :: 2 : x + 3 \end{array}\right\}.$

8. $\left\{\begin{array}{c} x + y : y + z :: 7 : 4 \\ x + z : x + y :: 5 : 7 \\ y + z : x + 4 :: 1 : 2 \end{array}\right\}.$

9. $\left\{\begin{array}{c} x + y : z :: 2 : 1 \\ x - 2 : y + z :: 2 : 3 \\ x + y + z = 12 \end{array}\right\}.$

10. $\left\{\begin{array}{c} x + y : 8 :: 4 : x - y \\ x + y = 8 \end{array}\right\}.$

11. $\left\{\begin{array}{c} 2 : y :: x : 24 \\ x - y : 4 :: 1 : x - y \end{array}\right\}.$

12. $x + 3 : 5 :: 8 : 2.$

13. $x - 2 : x - 4 :: 21 : x + 6.$

14. $2x : x + 3 :: 5 : 2x - 6.$

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192. The ratio of $4x : 5x$ is that of 4 to 5; for $\frac{4x}{5x} = \frac{4}{5}$. If there are two numbers in the ratio of 4 to 5, one number may be represented by $4x$ and the other by $5x$. We may, however, represent them by x and y, and then we will have the proportion $x : y :: 5 : 4$. The former method is generally preferable, because it uses only one letter.

PROBLEM. Two numbers are in the ratio of 3 to 5, and their sum is 24. What are the numbers?

STATEMENT. Let $3x$ and $5x$ represent the numbers, because they are in the ratio of 3 to 5.

And because their sum is 24, we have the equation,

$$3x + 5x = 24.$$

SOLUTION.

Collecting and dividing, $x = 3.$

$3x = 9,$ one number;

$5x = 15,$ the other number.

Proof. $9 + 15 = 24.$

PROBLEMS.

1. Divide 36 into two parts which will be in the ratio of 7 to 2.

2. Two boys had each the same number of marbles, but one lost 4 and the other bought 5, when their numbers were in the ratio of 2 to 5. How many had each at first?

3. In a game of cricket the side at the bat had 60 runs less than the other. They scored 10 more, when the score stood 2 to 1. How many had each at last?

4. A man leaves $18,000 to be divided between 3 persons in the ratio of 2, 3, and 4. What is the share of each?

5. In an alloy of gold, the gold was to the copper as 3 to 1, but 8 ounces of copper being added, the proportion was 7 to 5. How many ounces of pure gold were in the alloy?

6. A's age is to B's age as 4 to 5. 6 years ago their ages were as 10 to 13. What are their ages?

7. A's money is to B's money as 3 to 7. At the end of 6 months their gains are in the ratio of 5 to 3, A having then $1400, and B $2400. How much had each at first?

8. From 100 pounds of sea-water 92 pounds were boiled away, when the weight of the remaining pure water was to the weight of the salt it contained 11 to 5. How many pounds of salt were in the 100 pounds of sea-water?

9. On a farm the acres of woodland were to the acres of cleared land as 2 to 5, but 5 acres more having been cleared, the ratio became as 1 : 3. How many acres were in the farm?

SUGGESTIONS ON THE SOLUTION OF PROBLEMS.

The parts of these which are printed in *Italics* are meant for the oral explanations. The rest should be written down by the student, and the blanks occupied by stars should be filled by the proper expressions.

PROBLEMS ON PAGE 65.

1. Let x = number of years in son's age.
Then * * * = number of years in father's, *because he was* 3 *times as old as the son.*
* * * = number of years in son's age 6 years ago,
and * * * = number of years in father's age 6 years ago.

But because the father's age was then equal to 5 *times the son's, we form the equation* * * * *.

2. Let x = number of years in James' age.
Then * * * = number of years in John's age, *because he was* 6 *years older than James.*
Then * * * = number of years in James's age 3 years hence,
and * * * = number of years in John's age 3 years hence.

And because 3 *times John's age then would be equal to* 4 *times James's age at the same time, we form the equation* * * *.

3. Let x = number of hours the man will row after the boat.
Then * * * = number of hours the boat was drifting altogether, *because it drifted for* 6 *hours before he started,*
and * * * = number of miles the boat drifted,
and * * * = number of miles the man rowed.

But because the man rowed the same distance that the boat drifted, we have the equation * * *.

4. Let x = number of miles per hour the current moves.
Then * * * = number of miles he can row down stream in one hour,

and * * * = number of miles he can row up stream in one hour,

and * * * = number of miles he can row up stream in two hours.

And because the distance he can row down stream in one hour is equal to the distance he can row up stream in two, we form the equation * * *.

5. Let x = number of hours before A will overtake B.

Then * * * = number of miles A will walk in x hours at 4 miles an hour,

and * * * = number of miles B will walk in x hours at 3 miles an hour.

But because A walks 6 *miles more than B, since he goes and returns* 3 *miles, we can form the equation* (miles A walks equals miles B walks, plus 6).

6. Let x = number of days he worked.

Then * * * = number of days he was idle,

and * * * = number of dollars his wages would amount to, at $2 per day,

and * * * = number of dollars his board would amount to, at $1.25 per day.

And because he received $35, *that amount was the remainder after deducting his board from his wages. Hence we have the equation* (wages minus board equal money received).

7. Let x = number of dollars in one day's wages, then find expressions for one day's board, for 10 days' wages, and for 6 days' board, and form the equation as before.

ANSWERS.

Page 10.
1. 12.
2. 16.
3. 15.
4. 25.
5. 11.
6. 13.
7. 11.
8. 15.
9. 17.
10. 16.

Page 11.
1. 12.
2. 20.
3. 18.
4. 28.
5. 15.
6. 15.
7. 30.
8. 36.
9. 36.

Page 12.
1. 21.
2. 22.
3. 11.
4. 15.
5. 17.
6. 16.

1. 25.
2. 12.
3. 7.
4. 18.
5. 22.

Page 12.
6. 14.
7. 49.
8. 23.
9. 19.
10. 14.
11. 15.
12. 12.

Page 13.
1. $3a+2c$.
2. $a+3b$.
3. $2b+2c$.
4. $3a+3b+c$.
5. $3a+2b+2c$.
6. $3a+2b+2c$.
7. $2a+2b+2c$.
8. $a+2b+2c$.
9. $a+4b+c$.
10. $a+2b+3c$.
11. $2a+2b+c$.
12. $3a+3c$.

1. $3x$.
2. $3a+2y$.
3. $2b+2y$.
4. $3a+2x$.
5. $2a+b+2x$.
6. $b+3c+y$.
7. $2c+2x+y$.
8. $2a+2x+y$.
9. $2a+2m$.
10. $a+2c+2x$.
11. $2m+2n+2x$.
12. $a+3x+2y$.
13. $a+3b+2c$.

Page 13.
14. $2a+4x$.
15. $3b+3y$.
16. $3m+n+x$.
17. $4c+x+y$.
18. $3a+b+2y$.
19. $3x+3y$.
20. $3c+m+n$.
21. $2a+2m+x$.
22. $2a+c+m+x+y$.

Page 14.
1. $9a$.
2. $14b$.
3. $6x$.
4. $8c$.
5. $12y$.
6. $7m$.
7. $7c+2m$.
8. $6a+7c$.
9. $7b+9x$.
10. $11m+9y$.
11. $4c+2y$.
12. $7a+4b$.

1. $3a+5b$.
2. $7c+9x$.
3. $11b+5y$.
4. $7a+5b+2c$.
5. $11a+7x$.
6. $11m+6y$.

Page 16.
1. 4.
2. 3.

Page 16.
3. 3.
4. 7.
5. 5.
6. 4.
7. 5.
8. 2.
9. 6.
10. 2.
11. 5.
12. 4.
13. 2.
14. 3.
15. 2.
16. 3.
17. 9.
18. 6.
19. 6.
20. 4.

Page 17.
1. $2c$.
2. $4x$.
3. $5y$.
4. $2b$.
5. $2y$.
6. $6a$.
7. $5m$.
8. $3x$.

Page 18.
1. $2c$.
2. $4x$.
3. $3y$.
4. $5m$.
5. $4a$.
6. $4y$.

Page 18.

7. b.
8. $4y$.
9. $4b$.
10. $3a$.
11. $7y$.
12. $8m$.
13. $5b$.
14. $4a$.
15. $4c$.
16. $2y$.

1. $6a+b$.
2. $2a+2c$.
3. $7c-4b$.
4. $6b+c$.
5. $14b-5a$.
6. $3a-5b+4c$.
7. $6a+b$.
8. $13x+2y$.
9. $5a+9b$.
10. $5m+2x$.
11. $4a+2b$.
12. $6x+6y$.

Page 19.

1. 15.
2. 5.
3. 10.
4. 25.
5. 20.
6. 15.

1. 6.
2. 7.
3. 10.

Page 20.

4. 5.
5. 2.
6. 2.
7. 5.
8. 9.
9. 4.
10. 7.

1. 36.
2. 63.
3. 32.

Page 21.

1. 40.
2. 4.
3. 10.
4. 25.
5. 100.
6. 20.
7. 40.
8. 80.
9. 20.

1. a^2bc^2.
2. ab^2x^2.
3. a^2b^3x.
4. b^2cx^2.
5. bc^2x^3.
6. a^2m^2x.
7. $a^3b^2x^3y^2$.
8. $a^4b^2c^2x$.
9. $b^3c^2y^2$.
10. a^3m^4n.

Page 22.

1. 36.
2. 48.
3. 18.
4. 24.
5. 72.
6. 48.
7. 45.
8. 56.
9. 32.

Page 22.

10. 24.
11. 36.
12. 24.
13. 12.
14. 3.
15. 10.
16. 12.

1. $4b^2+2b$.
2. $ab+5b^2$.
3. $4ab+3b^2$.
4. $3abc+4b^2$.
5. $5ax+5ay$.
6. $5y^2+4y$.
7. $4x^2$.
8. $7a^2y$.
9. $3xy+2y^2$.
10. a^2+2b.
11. $4x^2+x$.
12. $4m^2+2m$.
13. $a^2-2b+4b^2$.
14. $2c^3+2ac^3$.

Page 24.

4. 3 yds. down.

Page 25.

5. 7 yds. up.

1. $-4y$.
2. $-2z$.
3. $5a$.
4. $-2ay$.
5. $2m^2$.
6. $4a^3$.
7. $-6amx$.
8. $3abc-3b$.
9. $4x^2y-2a$.
10. $2a-3b$.
11. $-3x^2-2ab$.
12. $3ab^2-5b^2c$.

Page 26.

1. $3a$.
2. $-2ax$.
3. $-ab$.
4. $-6m^2n$.
5. $5xy-4y^2$.
6. $2ax+x^2$.
7. $3a^2xy-3x^2y^2$

Page 27.

1. $6m^2+m$.
2. $8c^2d+4cd$.
3. $3y^3-9y$.
4. $2xz+8z^2$.
5. $3a^2y+3ay^2-2a^2y^2-4ay$.
6. $2x+2y$.
7. $2x^2+2x+3y^2$.
8. $7a^2+2ab-2bc-2b^2$.
9. $-3a+5c+7m$.
10. $3x^2+5x-4$.
11. $7m^2+7bm^2+2am$.
12. $b^2cx+2c^2x+2y^2$.
13. $2m^2+2mn-3nx$.
14. $14my^2-4m^2y-m^2-4a^2x-7$.
15. $3abc^2+3ab^2c-6a^2bc$.

Page 28.

16. $3m^3n+m^2n^2-mn^3$.

Page 28.

17. $6x^3-8x^2y-2y^3$.
18. $10b^2c^3x+b^3c^2-8b^4c-11x^2$.
19. $13a^3+14x^2-z-4c+3y$.
20. $-5abcx^2+10a^2bcx+8ab^3c$.

Page 30.

1. $3x$.
2. $4y$.
3. $-7x^2$.
4. $-3y$.
5. $7x^3$.
6. 0.
7. 0.
8. 0.
9. $3x$.
10. $-2b^2$.
11. $4y$.
12. x^2.
13. 0.
14. 0.
15. 0.
16. $-b^3$.
17. $8ay$.
18. $6x$.
19. $2ab$.
20. 0.
21. $-3ax$.
22. $8xy$.
23. $-5y^2$.
24. $10ab$.
25. $-7xy^2$.
26. $4x^2y$.
27. $-6abc$.
28. $-9ax^3$.

Page 31.

29. $-10xy$ and $4xy$.
30. $3x^2$ and $7x^2$.
31. $-6ab^2$ and $-8ab^2$.
32. $4bc^3$ and $-7bc^3$.
33. $-2x^3$ and $10x^3$.

Page 33.

1. $2ax^2$.
2. $-20am^2$.
3. $-11x^2y^4$.
4. $15x^3y$.
5. $5x^3y$.
6. $14abx^2$.
7. $18abc$.
8. $3ax$.
9. $-11x^6$.
10. $-2b^2c^3$.
11. $-3ax$.
12. $15a^3y$.
13. $3by^2-b^3$.
14. $2x^3y-7x^2y^2-2yz$.
15. $-3a^2+10b^2+3bc$.
16. c^3n-10n^3.
17. $-axy-7xy^2+2y^3$.
18. $5m^2+6mn-7mn^2$.
19. $-9b^2c+8bc^2+4mn$.
20. $6x^2$.

Page 34.

21. $a-5b+7c$.
22. $3x+3y$.
23. $5xy+9y^2-2z$.
24. $2ax+6x^2-xy$.
25. $ax-6ay+2x^2+3az$.

Page 34.

26. $-4by^2+2b^2y-3b^3-5b^4$.

Page 35.

1. $8z-4z=2-4+12-2$.
2. $5y+4x-2y-y=7$.
3. $2x-3x+8x=4+1+9$.
4. $3x+2y=-2+5$.
5. $6x-2x=-4+7+5$.
6. $4x+2x=-5+7+10$.

Page 36.

1. $y=5$.
2. $x=2$.
3. $x=3$.
4. $y=8$.
5. $y=3$.
6. $x=4$.
7. $x=24$.
8. $x=3$.
9. $x=8$.
10. $y=1$.
11. $x=7$.

Page 37.

1. 3.
2. 9.
3. 8.
4. 6.
5. −1.
6. 4.
7. 3.
8. 6.
9. 4.

Page 38.

10. 10.
11. 19.
12. 12.

Page 38.

13. 11.
14. 3.
15. 7.
16. 4.

Page 40.

1. $3x$.
2. $3x+2$.

Page 41.

3. $4x-5$.
4. B, $2x-4$; C, $2x+6$.
5. $x+1$; $x+2$.
6. $x-5$.
7. $x+3$.
8. $100-x$.
9. $2x+3$; $x-2$.
10. $3x+2$; $4x+2$.
11. $4x$; $3x+12$.

Page 43.

1. 5 and 13 inches.
2. 137 pages.
3. 126 and 234.

Page 43.

4. 50 and 98.
5. Corn, 40; wheat, 80.

Page 44.

6. A, $900; B, $300.
7. 55 feet.
8. J., 13; W., 31.
9. Wheat, 30; oats, 40; corn, 60.
10. 13 and 37 years.
11. 35 and 13.
12. J., 7; H., 15.
13. B, $500; A, $1000.
14. 13 feet.

Page 45.

15. Mars, 2; Jupiter, 4; Saturn, 8.

Page 48.

1. $21ay^2$.
2. $-30abc$.
3. $9a$.
4. $36bc^2y$.

Page 48.

5. $-24x^5$.
6. $48xy^3$.
7. $24ax+12x^2-18xy$.
8. $21b^2-12bc+24c^2$.
9. $25axy+30x^3y-40y^2$.
10. $-16acd+40c^2-80a^3$.

Page 49.

1. $24m^2n$.
2. $-27b^3c^2$.
3. $-55xy^2$.
4. $-12ab-9b^2+6c$.
5. $-32m^2+12mn-4n^2$.
6. $45ac-20cd+30ab$.
7. $-49x^2+35xy+28y^2$.
8. $-66b^3c+24b^2c^2-30bc^3$.

Page 50.

1. abc.
2. $6abcxy$.
3. $14amnx$.
4. $21acdmx$.
5. $8x^2y^2$.
6. $20a^3c$.

1. b^7.
2. c^6.
3. m^8.
4. ab^4.
5. a^3m^3.
6. $12c^5y^8$.
7. $32a^7$.
8. $-21x^2y^5$.

Page 50.

9. $-44m^4n^2x$.
10. $24a^3c^4x^6$.
11. $35ab^5c^5y^3$.
12. $-9abc^3x^3y^4$.
13. $-21ac^3d^6x$.
14. $-60m^8x^4y^2$.
15. $80a^3x^7y^{11}$.
16. $48a^4b^4c^3d$.
17. $30b^5c^4-24b^4c^3+12b^2c^8$.
18. $-35a^4x^3y^2-45a^5xy^4-5a^7y^{10}$.

Page 51.

19. $28a^3m^2n^6-24am^4n^5+8m^6n^4$.
20. $42c^6d^6+35c^5d^7-56c^4d^8$.
21. $72x^4y^3+36x^2y^5-45xy^4$.

Page 51.

22. $-80m^2x^3+40mx^4+8mx^2y$.
23. $-21xy^5+12xy^4-24xy^3$.
24. $54a^5by-30a^5xy+18a^3xy^2$.
25. $6a^2c^4x-5a^4c^2xy+13a^2c^2y^4$.
26. $13b^2m^5-8b^4m^3+5b^6m^2$.
27. $12a^3xyz+12a^2xyz^2-12a^2yz^3$.
28. $15xy^4+15x^4y-12x^2yz$.
29. $28c^3d^5y-36cd^4y+4d^2x^5y$.
30. $-45a^3m^2x^5+36a^3mx^6$
$-3a^3x^3y^2$.

Page 52.

1. $2ay+4y^2+4z$.
2. $-c^3d-2cd^3-4d^3x$.
3. $3a+7x$.
4. $7ay-4y^3$.
5. $5x^2y+2xy^2$.
6. $a^2+2ax+x^2$.
7. m^2-n^2.
8. $4c^2-x^2$.
9. $a^2-b^2-c^2$.
10. $x^4+x^2y^2+y^4$.
11. $8a^5+8a^4x-12a^3x^2+7a^2x^3$
$-2ax^4$.
12. $2a^2+5ax-x^2$.
13. $30c^2+17cd-21d^2$.
14. $2a^2-ab-3b^2$.
15. $x^6-x^5y-x^3y^3-xy^5-y^6$.

Page 54.

1. $a^2+2ax+x^2$.
2. $a^2-2ab+b^2$.
3. $2a^2-5ab+3b^2$.
4. $30c^2+17cd-21d^2$.
5. a^2-m^2.
6. $9c^2-16d^2$.
7. $a^2+2ab+b^2+ac+bc$.
8. $a^2-ab-2ac+bc+c^2$.
9. $10a^3+31a^2x+17ax^2+5x^3$.
10. a^3-x^3.

Page 54.

11. $3y^3-11y^2z+10yz^2-12z^3$.
12. $ac+bc-ad-bd$.
13. $21x^3-41x^2y+6xy^2+8y^3$.
14. $2x^3+2y^3$.
15. $x^3-3x^2y+3xy^2-y^3$.
16. $x^4-x^3y+2x^2y^2-xy^3+y^4$.
17. $4a^2-9b^2+6bc-c^2$.
18. $6a^4x+3a^3x^2+11a^2x^3+6ax^4$
$-2x^5$.
19. $81a^4-16c^4$.
20. $15a^5-2a^4-8a^3$.
21. $2c^4+7c^3m-3c^2m^2-cm^3$
$+m^4$.
22. a^4-b^4.
23. $a^3+5axy-6x^2y+a^2y+3xy^2$.
24. $30x^5-18x^4-15x^3+33x^2$
-12.
25. $a^2+ab+ax-ac+bx-cx$
$-ay-by+cy$.
26. $3x^6-8x^5+11x^4-16x^3+14x^2$
$-2x-2$.

Page 55.

27. a^4-b^4.
28. $a^4+a^3x-ax^3-x^4$.
29. $6c^4-23c^3+18c^2+12c-8$.
30. $3a^2x^4-3ax^3$.

Page 56.

1. $3m-2n-x$.
2. $4a+b+x-y$.
3. $6y-a-b$.
4. $2am-3mn+2m^2$.
5. $4ay-3ax-x^2$.
6. $9mx-2m^2+x^2$.
7. $3bc-2by-cx-3ab$.
8. $2y^2-3ay-2xy-4yz+z^2$.
9. $4ax+2x-y-3x^2+y^2$.

Page 57.

1. $6x+y$.
2. $4y^2-z$.
3. $a+b$.
4. $4x+3y$.
5. $-3c-9d$.
6. $3a^2-2a$.
7. $9a^2x+3a^2$.
8. $5xy-7y^2$.

1. $14x-21y+7z$.
2. $-12x+8y$.
3. $2a^2-2ab$.
4. $14b^3c-21bc^3$.
5. $-4a^3y+8a^2y^2$.
6. $12mx^3-6x^4$.
7. $-6a^3x^3+2ax^5$.
8. $-4x^2y^4+xy^5$.

Page 58.

1. $5-x-3=7+2x-2$.
2. $x+6-2x=4-2-x$.
3. $y-y+4=4y+24$.
4. $30-5x=x-2x+6$.
5. $15-28-4x=18x-27$.
6. $-24+8x=12-4x+12$.

1. $x=3$.
2. $x=5$.
3. $x=8$.

Page 59.

4. $x=6$.
5. $x=4$.
6. $x=8$.
7. $x=-13$.
8. $x=4$.
9. $x=7$.
10. $y=9$.
11. $x=5$.
12. $x=1$.
13. $x=6$.

1. $8-2x-x^2$.
2. $x^2-4x-21$.
3. $10x^2+34x-24$.
4. $a^2-ab-6b^2$.
5. $2x^2+11x-6$.
6. $12x^2-26x+10$.

Page 60.

1. $x=7$.
2. $x=4$.
3. $x=1$.
4. $x=3$.
5. $x=5$.
6. $x=1$.
7. $x=3$.
8. $x=3$.

Page 61.

9. $x=5$.
10. $x=6$.
11. $x=3$.
12. $x=52$.
13. $x=4$.
14. $x=11$.
15. $x=4$.
16. $x=4$.
17. $x=-6$.
18. $x=7$.
19. $x=3$.

Page 62.

1. $3x-5$, $6x-10$.
2. $x+6$.
3. $4(x+6)$.
4. $x+2$, $3(x+2)$.
5. $4(x-2)$.
6. $1500-50x$.
7. 3 times.
8. $5(x-5)$.
9. Length, $x+5$; contents, x^2+5x.
10. $x+5$ width; $x+4$ length; contents, $x^2+9x+20$.

Page 63.

1. $x=50$.
2. $x=500$.

Page 65.

1. Son, 12; father, 36.
2. James, 13; John, 19.
3. 4 hours.
4. 1 mile.
5. 6 hours.
6. 20 days.
7. Wages, $4; board, $2.50.
8. Children, 30 cents; women, 90 cents; men, $1.25.

Page 66.

9. 12 years.
10. Harness, $50; horse, $150; carriage, $200.
11. Elephant, 10 feet; giraffe, 16 feet.
12. 32 sheep.
13. 30 boys; 42 girls.
14. 12 by 15; 18 by 10.
15. 12 cents.
16. $1000.

Page 67.

17. \$1000.
18. 5 boys; 37 girls.
19. 3 miles.
20. \$6.

Page 68.

1. $x=5$.
2. $y=2$.
3. $x=6$.
4. $x=4$.

Page 71.

1. $x=2$; $y=4$.
2. $x=5$; $y=1$.
3. $x=7$; $y=3$.
4. $x=3$; $y=3$.
5. $x=2$; $y=12$.
6. $x=6$; $y=2$.

Page 73.

1. $x=4$; $y=2$.
2. $x=9$; $y=3$.
3. $x=1$; $y=7$.
4. $x=5$; $y=2$.
5. $x=3$; $y=-2$.
6. $x=-2$; $y=3$.
7. $x=8$; $y=4$.
8. $x=12$; $y=10$.
9. $x=1$; $y=4$.
10. $x=9$; $y=4$.

Page 74.

1. $x=9$; $y=5$.
2. $x=4$; $y=7$.
3. $x=1$; $y=6$.
4. $x=-4$; $y=5$.
5. $x=4$; $y=1$.
6. $x=10$; $y=1$.

1. 2 and 1.

Page 75.

2. 11 and 7.
3. 7 and 13.
4. Rice, 8 cents; sugar, 9 cents.
5. Coffee, 28; tea, 65.
6. Calico, 8; muslin, 10.
7. Cows, 30; calves, 6.

1. $40x$.

Page 76.

2. $10x+25y$.
3. $x+y=16$.
4. $100x+90y=1500$.
5. $x-10$.
6. $y+10$.
7. A, $x+6$; B, $y-6$.

1. Man, \$1.20; boy, .30.
2. 4 and 8.
3. Butter, 20; sugar, 9.

Page 77.

4. Goods, 15; eggs, 12.
5. Lead, 4; iron, 5.
6. A, 15; B, 12.
7. Chickens, 30; ducks, 40.
8. Corn, 4; oats, 8.
9. 24 at 14 cents, and 16 at 24 cents.
10. Muslin, 10 and 20 yards; linen, 20 and 40 cents.

Page 78.

11. Wages, \$3; board, \$1.25.
12. 7 and 5.

Page 78.

13. 180 shad; 2400 herring.
14. A, \$16; B, \$6.
15. At trade, 40; cutting wood, 8.
16. Men, \$8; boys, \$3.

Page 79.

1. $3a^2$.
2. $3abc$.
3. $3ax^2$.
4. $-3x^3$.
5. $3a^2y$.
6. $-4b^2$.

Page 80.

1. $-2ax^3$.
2. $10y^2z$.
3. $-2mn^2$.
4. $-4abc$.
5. $3c^3d$.
6. $-2ay^3$.
7. $-5a+4b-3c$.
8. $-10ax-6x^2+3xy$.
9. $-2x^2+4y^2+8z^2$.
10. $7c^2-6cd+3d^2$.
11. $-7b^2+8bc-3c^2$.
12. $8abc-10ab^2+2c^3$.

Page 81.

1. $7a^3-4b+5$.
2. $7b-8a+9x$.
3. $4bc-5x+10xz$.
4. $4a-3dx+1$.
5. $9ac-3dx+2y$.
6. $4b+7y-2$.

Page 82.

1. a^3.
2. $7ax^2$.

Page 82.

3. $-6bc$.
4. $5xy-6a$.
5. $2b-6e^2d$.
6. $2n-4m^2$.
7. $4c^2x-7a^2$.
8. $9x^2-10x+6$.
9. $-2ab+5x^2y$.
10. $-8b^2-2bc+c^2$.
11. $cx^2-ax^4+a^2cx^5$.
12. $an^2-bm^2y^2$.

Page 83.

13. $3b^4n^2-4a^3bn^4+2c^2n$.
14. $7x^2-4y^2+z^2$.
15. $2c^3d^2-5dy^2+3$.
16. $2n^2y-6b^3n+4n^3y^3$.
17. $9ax-12cx^4y^5+5a^2cx^2y$.
18. $2a^3cy-1+8x$.

1. $2x^3-3x^2+x$.
2. $y^4-5y^3+y^2-3y$.
3. a^3+7a^2-4a.
4. $b^6-3b^4+5b^2+b$.
5. $x^{10}-2x^8-4x^6+6x^4+x^2$.
6. $4y^5+6y^3-7y$.

Page 84.

7. $a^2+2ax+x^2$.
8. $x^3y^3+3x^2y+3xy^2$.
9. $c^3+c^2x+cx^2+x^3$.
10. $a+x$ and $a^3+3a^2x+3ax^2+x^3$.
11. $a+b$ and $a^2+2ab+b^2$.
12. $x^2+2xy+y^2$ and $x^3+3x^2y+3xy^2+y^3$.

Page 84.

13. a^2-ax+x^2 and a^3+x^3.
14. a^2-b and $a^4-2a^2b+b^2$.
15. $2x-2$ and $6x^3-2x^2-6x+2$.
16. a^2+ax+x^2 and $a^4+a^2x^2+x^4$.

Page 85.

1. $2y$.
2. $-2y$.
3. a^2x-ax^2.
4. $b-a$.
5. $x^2-x+3ax$.
6. a^2b-b^3.
7. $-a^2b-ab^2-b^3$.
8. $ab+b^2$.
9. $4a^2-ay+11y^2$.

4. Rem. $8x^2$.
5. Rem. 0.

Page 86.

6. Rem. 0.
7. Rem. $4x$.
8. Rem. $8x^2$.
9. $-17d$.
10. Rem. 16.
11. Rem. 0.
12. Rem. $ax+x^2$.
13. Rem. 0.
14. Rem. $xy+y^2$.
15. Rem. 0.
16. Rem. $-ab-b^2$.
17. Rem. 0.
18. Rem. $-a^2x-ax^2-x^3$.

Page 88.

1. 5.
2. $3+\frac{3b^2}{6ab+2b^2}$.
3. x.
4. y.
5. a.
6. $3a$.
7. x^2.
8. $2b+\frac{3b}{2ab+3}$.
9. $y+\frac{2y^2}{2x-y}$.
10. $2d+\frac{9d^2}{3c-4d}$.
11. 4.
12. 3.

Page 90.

1. $a+b$.
2. $c-d$.
3. $a+b$.
4. $x-y$.
5. $x^2-2xy+y^2$.
6. $2a+1$.

Page 91.

7. $x^3+x^2y+xy^2+y^3$.
8. $x-y$.
9. a^2-b.
10. $3x^2+2x-1$.
11. $3x-1$.
12. $4a^2-4ab-6b^2+\frac{-8b^3}{2a-2b}$.
13. $3c^2-2cd+d^2$.
14. $2a+3b-2c$.
15. $a^2+ax+x^2+\frac{2x^3}{a-x}$.
16. $a^3-a^2x+ax^2-x^3+\frac{2x^4}{a+x}$.

Page 91.

17. a^2-ax+x^2.
18. $x+y$.
19. m^3-2.

Page 93.

1. $m^2+2mn+n^2$.
2. $4+4x+x^2$.
3. b^2+6b+9.
4. $4a^2+4ab+b^2$.
5. $c^2+2cd+d^2$.
6. $16+8y+y^2$.
7. $a^2+2ax+x^2$.
8. $25x^2+20xy+4y^2$.

Page 94.

1. $a^2-2ab+b^2$.
2. $x^2-2xy+y^2$.
3. $m^2-2mn+n^2$.
4. $9-6x+x^2$.
5. $4-4y+y^2$.
6. $9a^2-6ab+b^2$.

1. c^2-d^2.
2. m^2-n^2.
3. a^4-b^2.
4. $9-x^2$.
5. y^4-16.
6. x^4-x^2.
7. $9a^2-4b^2$.
8. $25y^4-16z^4$.

Page 95.

1. $x^2+2xy+y^2$.
2. $9a^2+6ab+b^2$.
3. $4x^2-8xy+4y^2$.
4. $36-x^2$.
5. $16a^2-16ab+4b^2$.
6. $a^4+2a^2b^2+b^4$.
7. $x^4-2x^3+x^2$.
8. $25y^4-4z^4$.
9. $49-28b+4b^2$.
10. $16a^2+24ab+9b^2$.
11. $4a^2c^2-4ac^2d$
    $+c^2d^2$.
12. $16x^2y^2+8xy^3+y^4$.
13. $9x^2y^2-y^4$.
14. $4-x^2$.
15. $16-x^4$.
16. $m^2n^2-2cdmn$
    $+c^2d^2$.
17. $x^6+2x^5+x^4$.
18. $9a^2b^2c^4-12a^2bc^4$
    $+4a^2c^4$.
19. x^6-x^2.
20. $4a^2x^2-12ax^3$
    $+9x^4$.
21. $9x^2-24xy+16y^2$.
22. $a^2-a^2b^2$.
23. $a^2x^2-4ax^3+4x^4$.
24. $4a^2+20ab+25b^2$.
25. m^4-n^4.
26. x^6-y^6.
27. $a^6+2a^3x^3+x^6$.
28. $a^2x^2y^2+2ax^2y^2z$
    $+x^2y^2z^2$.
29. $16a^2b^2+16ab^3$
    $+4b^4$.
30. $m^2n^2-2mn^3+n^4$.
31. $4x^2-16y^2$.
32. $1-6x+9x^2$.
33. $16y^2+8y+1$.
34. $36a^2-1$.

Page 96.

1. $x+y$.
2. $2a+b$.
3. $3c+4d$.
4. $5x+2y$.
5. $b-c$.
6. $3m-n$.

Page 97.

7. $a+b$.
8. $2-x$.
9. $4y+5z$.
10. x^4-x^2.
11. $a-3b$.
12. $x-3$.
13. $ab-cd$.
14. $2m-n$.
15. $5x+y$.
16. $2x+ab$.
17. $7x+5y$.
18. m^2-n.
19. x^2-x.
20. x^3-x^2.
21. x^4-y^4.
22. $x-y$.

Page 98.

1. 2, 2.
2. 2, 3.
3. 3, 7.
4. 2, 2, 2.
5. 2, 2, 3.
6. a, b, c.
7. a, a, b.
8. c, c, c.
9. c, c, x, x.
10. x, x, y.
11. 2, 3, a, a, x.
12. 2, 2, 2, x, y, y.
13. 3, 5, m, m, n, n.
14. 2, 2, 3, d, d, d, d.
15. 2, 3, 3, a, a, a, x.

1. a.
2. x.
3. y.
4. a.
5. n.
6. x.

Page 98.

7. $3y^2$.
8. mn.
9. a^2.
10. ax.
11. x^2y.
12. $3b^3$.
13. $a, x+y+z$.
14. a^2b^2, a^2-b^2.
15. $2y,\ x+y$.
16. $4, a^2+1$.

Page 99.

1. $a(a+y)$.
2. $2(a+x)$.
3. $3d(c+d)$.
4. $4x(x-2y)$.
5. $ab(c-b)$.
6. $ax(a-3x)$.
7. $c(d+x+y)$.
8. $2a(a+b+x)$.
9. $a(a^2+2ab-b^2)$.
10. $y(3x^2-3xy+y^2)$.
11. $3a(2b-4c+3y)$.
12. $bc^2(a^2b+cxy)$.
13. $2xy^2z(4x^2-7z)$.
14. $4c^2d^2(3c-d)$.
15. $10axy(5x+7y)$.
16. $12m^2n^2(m+2n-5x)$.

Page 100.

1. $(c-d)(c-d)$.
2. $(c+1)(c+1)$.
3. $(2d-y)(2d-y)$.
4. $(m+n)(m+n)$.
5. Not a square.
6. Not a square.
7. $(2x+3y)(2x+3y)$.
8. $(5+x)(5+x)$.
9. Not a square.
10. $(x^2-2y)(x^2-2y)$.
11. $(a^2-a)(a^2-a)$.
12. $(x^3+x^2)(x^3+x^2)$.

Page 101.

13. $(2+xy)(2+xy)$.
14. $(4x-y)(4x-y)$.
15. Not a square.
16. Not a square.

1. $(c+d)(c-d)$.
2. $(a+b)(a-b)$.
3. $(m+1)(m-1)$.
4. $(a^2+b)(a^2-b)$.
5. $(x^3+y^3)(x^3-y^3)$.
6. $(ab+c)(ab-c)$.
7. $(m+nx)(m-nx)$.
8. $(1+x)(1-x)$.
9. $(3x+2y)(3x-2y)$.
10. $(4ab+x)(4ab-x)$.
11. $(5+a^2)(5-a^2)$.
12. $(a^3+b^4)(a^3-b^4)$.
13. $(a+7b)(a-7b)$.
14. $(xy^2+z^3)(xy^2-z^3)$.
15. $(2ab^3+5x^2)(2ab^3-5x^2)$.

1. $(b+x)(b+x)$.
2. $(ab+c)(ab+c)$.
3. $(2x-y)(2x-y)$.
4. $(3x+4y^2)(3x-4y^2)$.
5. $(x^3+y^3)(x^3-y^3)$.
6. Not a square.
7. $(x^4+y^3)(x^4-y^6)$.
8. Not a square.
9. Not a square.
10. $(a^5x+3y^2)(a^5x-3y^2)$.

Page 101.

11. Not a square.
12. Not a square.

Page 102.

1. $(a+b)(a^2-ab+b^2)$.
2. $(a-b)(a^2+ab+b^2)$.
3. $(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)$.
4. $(x-y)(x^6+x^5y+x^4y^2+x^3y^3+x^2y^4+xy^5+y^6)$.
5. $(ab+c)(a^4b^4-a^3b^3c+a^2b^2c^2-abc^3+c^4)$.
6. $(x+y)(x^2-xy+y^2)$.
7. $(cd-x)(c^2d^2+cdx+x^2)$.
8. $(1+x)(1-x+x^2)$.
9. $(a-1)(a^4+a^3+a^2+a+1)$.
10. $(1-b)(1+b+b^2)$.

Page 105.

1. $a(a+b)(a-b)$.
2. $2x(x-y)(x-y)$.
3. $3a(a+1)(a+1)$.
4. $axy(x-y^2)$.
5. $5c^2x(d+x)(d+x)$.
6. $(x+2)(x-2)$.
7. $3(x^2+1)(x+1)(x-1)$.
8. $2(2x-1)(2x-1)$.
9. $x^2(x+2)(x-2)$.
10. $5x^4(x+2)(x-2)$.
11. $(x+y)(x^2-xy+y^2)$.

Page 105.

12. $(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$.
13. $4ax(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$.
14. $2a(9a^2+12ab+b^2)$.
15. $5a(2a-1)(4a^2+2a+1)$.
16. $7a^2(a^4+x^4)(a^2+x^2)(a+x)(a-x)$.
17. $abxy(a^2b^2+x^2y^2)(ab+xy)(ab-xy)$.
18. $3c^3(c^2+1)(c^2+1)$.
19. $4a(a^2+ab+b^2)$.
20. $6(2+x^2)(2-x^2)$.

1. $6ax$.
2. x^5.
3. 5 and 7.
4.
5. 5 and 8.
6. 0.
7. 5.
8. 6.
9. 8.
10. 3.
11. -12.
12. 8.
13. 14.

Page 108.

1. a^2b^2.
2. x^2yz.
3. ab^2x^3y.
4. $m^3n^2x^2y^3$.
5. $a^2c^3d^2x^3$.

Page 106.

6. am^3nx^2y.
7. $a^2b^3xy^3$.
8. $am^2n^3y^4z^2$.
9. $a^2d^3x^3y^2$.
10. $a^2cxy^2z^3$.

1. 12.
2. 12.
3. 30.
4. 12.
5. 12.
6. 24.
7. $14ax^2y$.
8. $20ax^3$.
9. $18ax^2y^3$.
10. $6a^3x^2y$.
11. $12x^2y^3$.
12. $12a^4y^3$.

Page 109.

1. $ab(a+x)$.
2. $ax(x-y)(x+y)$.
3. $4(a-b)(a+b)$.
4. $ax(a-x)^3$.
5. $3(x+4)(x-5)$.
6. $4(x+6)^2$.
7. $10(a+y)^2(a-y)$.
8. $6a(x+y)^2(x-y)$.
9. $x(x+1)(x+2)$.
10. $(a+b)(a+c)(c+b)$.

Page 110.

1. $6a^2-6x^2$.
2. $4x^3y^2-4xy^4$.
3. $x^4+x^3y-xy^3-y^4$.
4. $c^4dx-c^3d^2x$.
5. $a^4-2a^2b^2+b^4$.
6. $a^6+a^5x-a^4x^2-2a^3x^3-a^2x^4+ax^5+x^6$.

Page 110.

7. $20b^4c-5b^2cy^2$.
8. $8a^3b-4a^2b^2$.
9. $18a^3xy^3+18a^2bxy^3-18ab^2xy^3-18b^3xy^3$.
10. $16c^5d^3+24c^4d^4+8c^3d^5$.
11. x^4-y^4.
12. $amnx^2y+amnxy^2$.

Page 111.

1. a^2-b^2.
2. x^2-y^2.
3. $a^4+2a^3b+a^2b^2$.
4. $3x^3-6x^2y+3xy^2$.
5. c^2-d^2.
6. a^4-ab^3.
7. x^4-y^4.
8. a^4-b^4.
9. $8a^2xy-8b^2xy$.

Page 113.

1. $\frac{a}{2n}$.
2. $\frac{3xy}{4az}$.
3. $\frac{1}{3acx}$.
4. $\frac{mn}{2ay}$.
5. $\frac{2d}{by}$.
6. $\frac{5bnz}{7xy}$.
7. $\frac{x}{a-b}$.
8. $\frac{a+y}{y(a-y)}$.
9. $\frac{3x+2}{2(4x-3)}$.

Page 113.

10. $\frac{a}{y}$.

Page 114.

1. $\frac{2c}{3d^2}$.
2. $\frac{3a}{4b^4}$.
3. $\frac{1}{2mn^3x^2}$.
4. $\frac{7cy^2}{4ax}$.
5. $\frac{2a^5y^3z}{bcx^2}$.
6. $\frac{cmn^3y^2}{2d^2x^3}$.
7. $\frac{2m^4n}{3a^3bx^3y^2}$.
8. $\frac{4}{9}$.
9. $\frac{3mx^4z}{5}$.
10. $\frac{1}{2bc}$.

Page 115.

1. $\frac{1}{x+y}$.
2. $\frac{x-y}{x+y}$.
3. $\frac{b}{a^2+b}$.
4. $\frac{1}{x+y}$.
5. $\frac{4}{5}$.
6. $\frac{7}{x+y}$.
7. $\frac{x+y}{x-y}$.
8. $\frac{a^2-ab}{ab+b^2}$.

Page 115.

9. $\frac{2a^2+3b^2-4ac^3}{6b^2+5c^3+2ax^2}$.
10. $\frac{3ax^2y^3-7ay^2}{8x^3z+2x^3y^3z^2}$.
11. $\frac{x^2+xy+y^2}{x-y}$.
12. $\frac{x-y}{x^2-xy+y^2}$.
13. $\frac{a-b}{a+b}$.
14. $\frac{2a+2x}{3a-3x}$.

Page 116.

1. $\frac{1}{x+y}$.
2. $\frac{n}{m^2-n^2}$.
3. $\frac{1}{c+d}$.
4. $\frac{a+b}{a-b}$.
5. $\frac{a+x}{a-x}$.
6. $\frac{3}{8}$.
7. $\frac{a+b+c}{x+y+z}$.
8. $\frac{2}{x-y}$.
9. $\frac{c+d}{c}$.
10. $\frac{x}{x+y}$.
11. $\frac{m}{x}$.
12. $\frac{a+1}{x+y}$.

Page 117.

1. $x+y$.
2. $a+b+\frac{b^2}{a}$.
3. $2c+3d+\frac{4d^2}{c}$.
4. $\frac{x}{y}+1$.
5. $1-\frac{d}{c}$.
6. $x+1+\frac{1}{x}$.
7. $5a+7y+y^2$.
8. $\frac{x^2}{y}+x+y$.
9. $3a+6ac+4d$.
10. $1+\frac{b}{a}+\frac{c^2}{a^2}$.

Page 118.

1. x^2+3xy.
2. $3b+\frac{-3ab^2+3b^3}{a^2-b^2}$.
3. $2c+\frac{4cd-8d^2}{3c-2d}$.
4. $x^4+x^3y+x^2y^2$
   $+xy^3+y^4$.
5. $a^5-a^4b+a^3b^2$
   $-a^2b^3+ab^4-b^5$
   $+\frac{2b^6}{a+b}$.
6. x^2-xy+y^2
   $-\frac{2y^3}{x+y}$.
7. $c+d-\frac{2d^3}{c^2-cd+d^2}$.
8. $a^2-7ab+8b^2$
   $+\frac{-21ab^3+24b^4}{a^2-3b^2}$.
9. $x-\frac{y^2}{x+y}$.
10. $x+\frac{y^2}{x-y}$.

Page 118.

1. $\frac{5x^3y}{xy}$.
2. $\frac{2a^2b+2ab^2}{2ab}$.
3. $\frac{4x^4y-4x^3y^2}{x-y}$.

Page 119.

4. $\frac{x^2-y^2}{x+y}$.
5. $\frac{a^2+2ab+b^2}{a+b}$.
6. $\frac{4m^3n+8m^2n^2+4mn^3}{4mn}$.

1. $\frac{cx+ab}{x}$.
2. $\frac{m^2n+n^2}{m}$.
3. $\frac{a^2+ab+c^2}{a}$.
4. $\frac{a^2-x^2}{a}$.
5. $\frac{bx-cx-y^2}{x}$.
6. $\frac{ax+x^2+y^2}{a}$.
7. $\frac{7x}{4}$.
8. $\frac{6y^2}{5}$.
9. $\frac{a^2+ab+x^2}{a+b}$.
10. $\frac{a^2+b^2}{a^2}$.

Page 120.

1. $\frac{2a^2+ab+b^2}{a}$.
2. $\frac{2x^2+3xy-3y^2}{x}$.
3. $\frac{2ab}{a+b}$.
4. $\frac{14c^2-12cd-3d^2}{3c}$.

Page 121.

5. $\frac{x^2-4ax-4a^2-6ay-3xy}{2a+x}$.
6. $\frac{ax+bx+ay+by-c-d}{x+y}$.
7. 0.
8. $\frac{-2a^2b+2ab^2+b^3}{a-b}$.

1. $\frac{m^2}{m+n}$.
2. $\frac{-b^2}{a-b}$.
3. $\frac{x^2-4xy+y^2}{x-y}$.
4. $\frac{2ab}{a+b}$.
5. $\frac{8c+6d}{5}$.
6. $\frac{3a+19b}{3}$.
7. $\frac{y}{x+y}$.
8. $\frac{b}{a-b}$.
9. $\frac{m}{m-n}$.
10. $\frac{x}{y}$.

Page 123.

1. $\frac{x^6}{3x^2}, \frac{3xy}{3x^2}, \frac{3a}{3x^2}$.
2. $\frac{ab-ac}{b^2-c^2}, \frac{b^2+bc}{b^2-c^2}$.

Page 123.

3. $\frac{ac-ad}{c^2-d^2}$, $\frac{bc+bd}{c^2-d^2}$, $\frac{c^2}{c^2-d^2}$.

Page 124.

4. $\frac{m^3n-mn^3}{m^2n^2-n^4}$, $\frac{am^3-amn^2}{m^2n^2-n^4}$, $\frac{mn^2x-n^3x}{m^2n^2-n^4}$, $\frac{mn^2y+n^3y}{m^2n^2-n^4}$.

5. $\frac{40\,a}{120}$, $\frac{20\,b}{120}$, $\frac{15\,c}{120}$, $\frac{12\,a}{120}$.

6. $\frac{a(a-b)^2}{6(a-b)^3}$, $\frac{6\,b^2(a-b)}{6(a-b)^3}$, $\frac{6\,a^2b}{6\,(a-b)^3}$.

7. $\frac{7\,a^2c+14\,abc+7\,b^2c}{35\,a^2+70\,ab+35\,b^2}$, $\frac{5\,a^3+10\,a^2b+5\,ab^2}{35\,a^2+70\,ab+35\,b^2}$, $\frac{35\,ab+35\,b^2}{35\,a^2+70\,ab+35\,b^2}$, $\frac{35\,a^2}{35\,a^2+70\,ab+35\,b^2}$.

8. $\frac{2\,x^2-2\,y^2}{x^3-xy^2}$, $\frac{x^2y-xy^2}{x^3-xy^2}$, $\frac{xy^2}{x^3-xy^2}$.

9. $\frac{a^4+2\,a^3b+a^2b^2}{a^4-2\,a^2b^2+b^4}$, $\frac{a^3b-2\,a^2b^2+ab^3}{a^4-2\,a^2b^2+b^4}$, $\frac{a^2b^2-b^4}{a^4-2\,a^2b^2+b^4}$.

10. $\frac{4\,x^2-4\,y^2}{x^5-x^3y^2}$, $\frac{5\,x^3}{x^5-x^3y^2}$, $\frac{8\,x^4+8\,x^3y}{x^5-x^3y^2}$.

1. $\frac{acd}{bcd}$, $\frac{b^2d}{bcd}$, $\frac{bc^2}{bcd}$.

Page 124.

2. $\frac{ax^2}{x^3}$, $\frac{bx}{x^3}$, $\frac{c}{x^3}$.

3. $\frac{a-b}{a^2-b^2}$, $\frac{a+b}{a^2-b^2}$.

4. $\frac{cx^3}{ax^2y^2}$, $\frac{cy^3}{ax^2y^2}$, $\frac{x^3y^3}{ax^2y^2}$.

5. $\frac{x^2}{x^2-y^2}$, $\frac{x-y}{x^2-y^2}$, $\frac{x+y}{x^2-y^2}$.

6. $\frac{c^2-2\,cd+d^2}{c^2-d^2}$, $\frac{c^2}{c^2-d^2}$.

7. $\frac{4\,y}{y}$, $\frac{x}{y}$.

8. $\frac{6}{x^2-2\,xy+y^2}$, $\frac{2\,x-2\,y}{x^2-2\,xy+y^2}$.

9. $\frac{a^2-2\,ax+x^2}{a^2-x^2}$, $\frac{a^2+2\,ax+x^2}{a^2-x^2}$.

10. $\frac{2\,x+2\,y}{xy^2+y^3}$, $\frac{ay^2}{xy^2+y^3}$.

Page 125.

1. $\frac{19}{12}$.

2. $\frac{41\,x}{24}$.

Page 126.

3. $\frac{abd+2\,bc+6\,d}{2\,bd}$.

4. $\frac{ax^2+bx+c}{x^3}$.

5. $\frac{7\,a+3\,b}{12}$.

6. $\frac{2\,x^2+2\,y^2}{x^2-y^2}$.

7. $\frac{c^2+d^2}{cd-d^2}$.

8. $\frac{3\,x}{x^2-y^2}$.

Page 126.

9. $\frac{2x^3+2x^2y+2xy^2+y^3}{x^3+3x^2y+3xy^2+x^3}$.

10. $\frac{ax^2-axy+b^2x+cx^2-cy^2}{x^3-xy^2}$.

11. $\frac{a^2+4a-8}{a^2-2a-8}$.

12. $\frac{c^3d+a^2b^2+abd^2}{a^2bc^2d}$.

13. $\frac{m^4+2m^3n+2m^2n^2-mn^3-n^4}{m^4-m^2n^2}$.

14. $\frac{2a^2+ab+b^2}{a^3-b^3}$.

1. $\frac{ay+bx}{xy}$.

2. $\frac{cd+b^2}{d^2}$.

3. $\frac{2x+y}{x^2+xy}$.

4. $\frac{2a}{a^2-b^2}$.

5. $\frac{4a+4c}{6}$.

6. $\frac{2a+b}{a^2-b^2}$.

7. $\frac{cy+dx+a^2}{axy}$.

8. $\frac{xz+x+y}{xz+yz}$.

9. $\frac{2a}{a^2-x^2}$.

10. $\frac{2x}{x^2-1}$.

11. $\frac{x^2+y^2}{xy}$.

12. $\frac{a+b+c}{a+b+c}=1$.

Page 126.

13. $\frac{2x}{x^2}\ \left(=\frac{2}{x}\right)$.

14. $\frac{x-x^2+y+xy}{xy}$.

Page 127.

1. $\frac{1}{15}$.

2. $\frac{3x}{8}$.

3. $\frac{a^3-b^3}{ab}$.

4. $\frac{4cd}{c^2-d^2}$.

5. $\frac{-2y}{x^2+2xy+y^2}$.

6. $\frac{-x^2+2xy+y^2}{xy}$.

7. $\frac{2+2a-b}{12}$.

8. $\frac{5xy-3x^2+2y^2}{x^2-y^2}$.

9. $\frac{2xy}{x^2-y^2}$.

10. $\frac{3ab}{a^2-b^2}$.

1. $\frac{b-a}{ab}$.

2. $\frac{bx-ay}{by}$.

3. $\frac{x+y-a}{ax+ay}$.

4. $\frac{2y}{3}$.

5. $\frac{c^2+cd-x^2+xy}{cx}$.

6. $\frac{ax-bx-by}{x^2+xy}$.

Page 128.

1. $\frac{a^2+4ab+b^2}{a^2-b^2}$.
2. $\frac{x^3-x^2y-y^3}{xy^2}$.
3. $\frac{b^4-2b^3c^2-2bc^3-2c^4}{b^2c^2-c^4}$.
4. $\frac{-x^3+16x^2+82x-100}{5x^2-5x-60.}$.
5. $\frac{3x^2-3xy}{x^2-y^2}$.
6. $\frac{2x^2+2ax-2a^2}{a^2x-x^3}$.
7. $\frac{a^2+b^2}{a^2-b^2}$.
8. $\frac{x^2+8x-4}{2x^2-8}$.
9. $\frac{4x^2-2xy-y^2}{x^3-2x^2y+xy^2}$.
10. $\frac{a^2+2a}{a^2-1}$.
11. $\frac{2x+1}{4-x^2}$.
12. $\frac{a^2+3ax+4ab-3bx}{x^2-a^2}$.

Page 129.

1. $3xy+\frac{2x-4xy}{x^2-y^2}$.
2. $c^2+\frac{2b}{a^2-b^2}$.
3. $2x+y+3z+\frac{bx^2+a^2y}{by}$.
4. $a-x-y+\frac{x-5y}{12}$.

1. $\frac{x^2+2xy+y^2}{xy}$.
2. $\frac{-a^3+2a^2b+b^3}{ab}$.

Page 130.

3. $\frac{25x}{12}$.
4. $\frac{x-3y}{x+y}$.

Page 131.

1. $\frac{ax}{by}$.
2. $\frac{ab}{d}$.
3. $\frac{ax}{b}$.
4. $\frac{a}{b}$.
5. $\frac{cm}{dn}$.
6. a^2c^2.

Page 132.

7. $\frac{cd}{a}$.
8. $\frac{4y^2}{x^3z}$.
9. $\frac{4y}{b^2}$.
10. $\frac{d}{a}$.
11. $\frac{4ax}{3}$.
12. $14y$.
13. $10x$.
14. $9x$.
15. 8.

1. $\frac{ax+ay}{ab^2+b^3}$.
2. $\frac{x-y}{xy}$.

Page 132.

3. $\frac{1}{a+b}$.
4. $\frac{ac-bc}{d}$.
5. $\frac{6-3x}{2}$.
6. x^4-2x^3y
   $+x^2y^2$.
7. $\frac{3ab}{4cd}$.
8. a^2-ax.
9. $\frac{2x-2y}{x}$.
10. $\frac{a+x}{a-x}$.

Page 133.

1. $\frac{x-y}{b}$.
2. $\frac{a+b}{a-b}$.
3. $\frac{x+y}{x-y}$.
4. $\frac{a}{b}$.
5. $3x$.
6. $\frac{y^2}{x^2}$.
7. $\frac{x-y}{x+y}$.
8. $\frac{a-b}{b}$.
9. $\frac{2+y}{2}$.
10. $\frac{c}{b}$.
11. $8+2x$.

Page 133.

12. $8x-24$.
13. $3x$.
14. $16x$.
15. $x-2$.
16. $8-4x$.
17. $\frac{6x-15}{4}$.
18. $\frac{x+7}{3}$.
19. $\frac{3x-5}{2}$.
20. $\frac{2+4x}{3}$.
21. $\frac{3x+4}{15}$.
22. $\frac{2+x}{6}$.

Page 134.

1. $\frac{ad}{bc}$.
2. $\frac{ay+by}{cx+ax}$.
3. $\frac{x}{x-y}$.
4. 1.
5. $\frac{y}{x}$.
6. $\frac{a+x}{a-x}$.
7. $\frac{x-y}{x+y}$.
8. $\frac{a^2y-axy+x^2y}{x}$.
9. $\frac{1}{x+y}$.

Page 134.

10. $\frac{1}{a+b}$.
11. $\frac{a^2+4x^2}{a^2+4ax+4x^2}$.
12. $\frac{x^2+10x+25}{x^2-2x+1}$.
13. $\frac{x^3}{y^3}$.
14. $\frac{2x^3}{y^3}$.
15. $\frac{y+x}{y}$.
16. $\frac{x+y}{2}$.

Page 135.

1. $\frac{bx}{ay}$.
2. $\frac{3y}{ax}$.
3. $\frac{a^2-b^2}{ac}$.
4. $\frac{4ax}{4xy+y^2}$.
5. $\frac{ay}{cx+cy}$.
6. $a+x$.
7. $\frac{m^2n}{x^2y-xy^2}$.
8. $\frac{14b}{3a+3b}$.
9. $\frac{1}{x-1}$.
10. $\frac{a+b}{b}$.
11. $\frac{2ac+2c^2}{a^2-2ac+c^2}$.

Page 135.

12. $\frac{x^2y^2-2xy^3+y^4}{x^3+x^2y}$.
13. $\frac{a^2+ab}{b}$.
14. $\frac{a-b}{a+b}$.
15. $\frac{a+x}{a-x}$.

Page 136.

1. $\frac{3a}{2x}+\frac{12}{x}+18$.
2. $\frac{8x}{3}+5y+36$.
3. $6x+5y$.
4. $28+6x+35x$.
5. $14+3+20x$.
6. x^2+2x+3.
7. $2x+6+40x$.
8. $6x-3+8x+6$.
9. $5+3x+9$.

Page 137.

10. $x+2+x-1$.
11. $8x-3-x$.
12. $40x+2x+4y$.
13. $24y-12y+4x$.
14. $2x+2y-x+y$.
15. $8x-x-1+\frac{2x-4}{3}$.
16. $2x+1-2x+1$.
17. $3x-9-4x$.
18. $a^2-2ab+b^2-a^2$
    $-2ab-b^2$.

Page 141.

1. $x=4$.
2. $x=10$.

Page 141.

3. $x=4$.
4. $x=8$.
5. $x=13$.
6. $x=5$.
7. $x=6$.
8. $x=2$.
9. $x=2$.
10. $x=9$.
11. $x=4$.
12. $x=12$.
13. $x=5$.
14. $x=2$.
15. $x=22$.
16. $x=12$.

Page 142.

17. $x=7$.
18. $x=3$.
19. $x=3$.
20. $x=9$.
21. $x=5$.
22. $x=7$.
23. $x=12$.
24. $x=2$.
25. $x=3$.
26. $x=3$.
27. $x=\frac{8}{7}$.
28. $x=4$.

Page 143.

1. $x=2$.
2. $x=2$.
3. $x=4$.
4. $x=9$.
5. 20.
6. 28.
7. 15.
8. 54, 55, 56.
9. 6, 8, 10.

Page 143.

10. 8.
11. 21.
12. 9.

Page 146.

1. John, 16; James, 14.
2. 15.
3. 12.

Page 147.

4. A, 28; B, 20.
5. Father, 60; son, 40.
6. John Adams, 41 and 91; Thomas Jefferson, 33 and 83.
7. 40 cents.
8. 12 in the first, 24 in the second, 30 in the third.
9. B, \$200; A, \$800.
10. \$300.
11. \$6.
12. \$140.

Page 148.

13. 96 miles.
14. Walk, $3\frac{1}{2}$; ride, 7.
15. 3 miles an hour, and 6 miles an hour.
16. Sailing vessel, 100; steamer, 300.
17. \$1200.
18. \$400.

Page 149.

19. \$11000.
20. \$75.

Page 149.

21. $3\frac{1}{2}$ and $4\frac{1}{2}$ miles per hour.
22. $\frac{4}{14}$.
23. $\frac{5}{6}$.
24. 1234.
25. \$100.
26. A, 2 hours; B, 4 hours.

Page 151.

1. $x=7$; $y=2$.
2. $x=8$; $y=5$.

Page 152.

1. $x=2$; $y=4$.
2. $x=5$; $y=7$.

Page 153.

3. $x=9$; $y=4$.
4. $x=6$; $y=10$.
5. $x=5$; $y=9$.
6. $x=3$; $y=8$.

Page 154.

1. $x=4$; $y=-1$.
2. $x=7$; $y=8$.
3. $x=3$; $y=\frac{1}{2}$.
4. $x=12$; $y=4$.
5. $x=-4$; $y=5$.
6. $x=10$; $y=4$.

Page 155.

1. $x=4$; $y=-5$.
2. $x=7$; $y=2$.
3. $x=6$; $y=1$.
4. $x=9$; $y=4$.

Page 156.

1. $x=13$; $y=4$.
2. $x=7$; $y=\frac{1}{3}$.
3. $x=5$; $y=6$.

Page 156.

4. $x=8$; $y=-4$.
5. $x=2$; $y=6$.
6. $x=\frac{3}{4}$; $y=2$.

1. $x=12$; $y=2$.
2. $x=5$; $y=-5$.
3. $x=7$; $y=\frac{5}{8}$.
4. $x=20$; $y=8$.

Page 157.

1. $x=8$; $y=2$.
2. $x=8$; $y=-2$.
3. $x=4$; $y=5$.
4. $x=9$; $y=-4$.
5. $x=6$; $y=4$.
6. $x=6$; $y=8$.

Page 158.

7. $x=7$; $y=2$.
8. $x=-2$; $y=8$.
9. $x=5$; $y=-7$.
10. $x=8$; $y=1$.
11. $x=4$; $y=3$.
12. $x=11$; $y=8$.
13. $x=\frac{1}{4}$; $y=\frac{1}{5}$.

Page 159.

1. $x=5$; $y=-25$.
2. $x=8$; $y=2$.
3. $x=2$; $y=-6$.
4. $x=\frac{1}{2}$; $y=\frac{1}{4}$.
5. $x=4$; $y=6$.
6. $x=4$; $y=\frac{1}{3}$.

Page 162.

1. $x=4$; $y=3$; $z=1$.
2. $x=2$; $y=5$; $z=4$.
3. $x=8$; $y=6$; $z=5$.

Page 162.

4. $x=4$; $y=5$; $z=2$.
5. $x=6$; $y=4$; $z=2$.

Page 163.

1. $x=5$; $y=4$; $z=2$.
2. $x=10$; $y=7$; $z=3$.

1. $x=4$; $y=7$; $z=2$.
2. $x=10$; $y=9$; $z=8$.

Page 164.

3. $x=8$; $y=6$; $z=3$.
4. $x=12$; $y=10$; $z=3$.
5. $x=9$; $y=4$; $z=2$.
6. $x=4$; $y=7$; $z=3$.
7. $x=8$; $y=2$; $z=3$.
8. $x=7$; $y=2$; $z=1$.
9. $x=5$; $y=6$; $z=2$.
10. $x=6$; $y=2$; $z=3$.
11. $x=7$; $y=2$; $z=1$.

Page 168.

1. $x=3$; $y=1$; $z=4$; $u=2$.
2. $x=5$; $y=4$; $z=3$; $u=2$.

Page 168.

3. $x=8$; $y=3$; $z=2$; $u=1$.
4. $x=3$; $y=4$; $z=2$; $u=1$.
5. $x=7$; $y=2$; $z=4$; $u=3$.
6. $x=\frac{1}{2}$; $y=\frac{1}{4}$; $z=\frac{1}{6}$.
7. $x=4$; $y=6$; $z=8$.

Page 170.

1. A, 16; B, 21.
2. 20 and 32.

Page 171.

3. 56 and 40.
4. 82.
5. $\frac{5}{7}$.
6. A, 18; B, 9.
7. Father, 10; son, 15.
8. $\frac{3}{5}$.
9. 42.
10. 36.
11. A, 25; B, $16\frac{2}{3}$.

Page 172.

12. 40 and 32.
13. 12.
14. 42 sheep; 2 oxen.
15. 20 and 80 hours.
16. 4 tops; 24 cents.
17. 65.
18. B, \$800; A, \$600.
19. Chickens, 40; geese, 80.

Page 173.

20. 8 cows; 30 sheep.

Page 173.

1. 15, 12, 10.
2. 40, 54, 36.
3. Wrist, 8; hand, 5; fingers, 14.
4. Height of monument, 220 feet; height of pyramid, 480; base of pyramid, 764.
5. A, 12 days; B, 15 days; C, 20 days.

Page 174.

6. Man, 42; sons, 70 and 105.
7. 459.
8. European, 94; Bushmen, 77; gorilla, 29.
9. A, $500; B, $700; C, $400.
10. 32 and 24.

Page 175.

1. Tail, 10; head, 22; body, 63; whale, 95.

Page 176.

2. Muslin, 10 cents; linen, 30 cents.
3. 9 quarts.
4. $\frac{2}{3}$ and $\frac{5}{8}$.
5. 20, 16, 18.
6. 30 gallons.
7. 1 pound saltpetre; 1 pound sulphur; 8 pounds charcoal.
8. $\frac{2}{3}$ of a month.

Page 177.

9. Voyage out, 70; at West Indies, 96; return voyage, 58.
10. $4\frac{1}{2}$ of heavy oats; $1\frac{1}{2}$ of light oats.
11. 40 and 51 pounds.
12. Man, $2\frac{2}{3}$ miles; son, $1\frac{1}{3}$ miles.
13. 30 cents.
14. Height of tower, 60 feet; track above water, 245 feet; length of bridge, 821 feet.

Page 180.

1. a^5.
2. 49.
3. 27.
4. x^6.
5. a^6x^2.
6. c^6d^9.

Page 181.

1. a^6b^3.
2. $a^8x^{12}y^4$.
3. $c^{14}d^{35}x^7$.
4. $a^{20}y^{10}z^5$.
5. $x^{12}y^8$.
6. $a^3c^6d^9x^{12}y^{15}$.
7. $c^2m^8n^2x^{10}$.
8. $b^6d^3m^{30}x^3$.
9. $a^8c^{16}x^{32}y^8$.

Page 182.

1. $9x^6y^4$.
2. $-125m^9n^{15}$.
3. $8a^9c^{15}$.
4. $16x^{16}z^{36}$.
5. $36a^2x^6y^2$.

Page 182.

6. $- \cdot x^3y^6z^{12}$.
7. $10000a^8x^{16}y^{12}$.
8. $16m^8n^{10}y^4$.
9. $81a^6c^2x^4$.

1. $3a^4x^3$.
2. $4a^7y^6$.
3. $3a^4x^6$.
4. $72a^3$.
5. $216x^7y$.
6. $5a^{12}x^9$.
7. $36a^4b^4x^4y^4$.
8. $4x^7y^5$.

1. $\frac{a^4}{4b^2}$.
2. $-\frac{x^9y^3}{27a^3b^6}$.
3. $\frac{16b^4x^2}{a^2y^6}$.
4. $\frac{a^4x^8y^4}{m^8z^{16}}$.
5. $\frac{c^8d^{12}}{x^4y^{16}}$.
6. $\frac{a^{12}x^4}{b^8d^{16}y^4}$.
7. $\frac{4x^6}{9a^4y^8}$.
8. $\frac{49a^2x^6}{a^2c^4d^2}$.
9. $\frac{1}{27a^3x^9}$.

Page 183.

1. $a^2+2ab+b^2$.
2. $a^2-2ax+x^2$.
3. $m^2+2mn+n^2$.
4. $x^2-4xy+4y^2$.

Page 183.

5. $16+8x+x^2$.
6. $9m^2+12mn+4n^2$.
7. $25a^2-60ab+36b^2$.
8. $9a^2x^2+12ax^3+4x^4$.
9. $16-24y+9y^2$.
10. $4x^2-4x^3+x^4$.
11. $25y^6-20y^5+4y^4$.
12. $9x^4+6x^2y+y^2$.

Page 183.

1. $x^3-3x^2y+3xy^2-y^3$.
2. $27a^3-54a^2b+36ab^2-8b^3$.
3. $a^4+8a^3x+24a^2x^2+32ax^3+16x^4$.
4. $64x^3-240x^2y+300xy^2-125y^3$.

Page 187.

1. $x^3+3x^2y+3xy^2+y^3$.
2. $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$.
3. $c^7+7c^6d+21c^5d^2+35c^4d^3+35c^3d^4+21c^2d^5+7cd^6+d^7$.
4. $m^8-8m^7n+28m^6n^2-56m^5n^3+70m^4n^4-56m^3n^5+28m^2n^6-8mn^7+n^8$.
5. $b^6+6b^5c+15b^4c^2+20b^3c^3+15b^2c^4+6bc^5+c^6$.
6. $a^5+5a^4c+10a^3c^2+10a^2c^3+5ac^4+c^5$.
7. $c^8+8c^7d+28c^6d^2+56c^5d^3+70c^4d^4+56c^3d^5+28c^2d^6+8cd^7+d^8$.
8. $m^6+6m^5n+15m^4n^2+20m^3n^3+15m^2n^4+6mn^5+n^6$.
9. $a^4-4a^3x+6a^2x^2-4ax^3+x^4$.

1. $x^3-3x^2y+3xy^2-y^3$.
2. $a^4+4a^3b+6a^2b^2+4ab^3+b^4$.
3. $c^3+3c^2d+3cd^2+d^3$.
4. $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$.
5. $m^4-4m^3n+6m^2n^2-4mn^3+n^4$.
6. $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$.

Page 188.

1. $8a^3+12a^2b+6ab^2+b^3$.
2. $81x^4-108x^3y+54x^2y^2-12xy^3+y^4$.
3. $c^5+10c^4d+40c^3d^2+80c^2d^3+80cd^4+32d^5$.
4. $x^{12}+6x^{10}y^2+15x^8y^4+20x^6y^6+15x^4y^8+6x^2y^{10}+y^{12}$.
5. $256y^4+256y^3z+96y^2z^2+16yz^3+z^4$.
6. $y^8+4y^7+6y^6+4y^5+y^4$.
7. $\frac{1}{8}x^6-\frac{3}{2}x^5+6x^4-8x^3$.
8. $a^8x^4+4a^6x^4+6a^4x^4+4a^2x^4+x^4$.
9. $x^5+\frac{5x^4y}{4}+\frac{5x^3y^2}{8}+\frac{5x^2y^3}{32}+\frac{5xy^4}{256}+\frac{y^5}{1024}$.

Page 188.

1. $x^3 + 3x^2y + 3xy^2 + y^3$.

2. $a^3 - 3a^2x + 3ax^2 - x^3$.

3. $m^3 + 3m^2n + 3mn^2 + n^3$.

Page 190.

1. 3.

2. 4.

3. $7x^2y$.

4. $2a^2c$.

Page 191.

5. $8a^2mx^4$.

6. $9xy^3$.

7. c^2d^6.

8. $3bx^2y^4$.

Page 192.

1. ± 3.

2. $\pm 2a$.

3. $2x^2$.

4. $\pm x^2y$.

5. $-m^2n^3$.

6. $\pm 2a^2b^3$.

7. $\pm 9a^2x^3y$.

8. $-3c^3d^4x$.

9. $2x^4y^5$.

10. $2a^8b^3c^2$.

Page 194.

1. $x-y$.

2. $3a+4$.

3. $c-5d$.

4. $ax-2$.

5. a^3+a^2+a.

6. x^2+2x-1.

7. a^2-2a+x.

8. b^3-2b+3.

9. a^2-ax-x^2.

10. $5x^2-2x+3$.

Page 195.

11. $x+y-z$.

12. $c+d+m$.

13. $a^2-3ax-x^2$.

14. $ax+ay+y^2$.

15. x^3-2x^2+3x-2.

Page 196.

1. $x=\pm 5$.

2. $y=\pm 2$.

3. $y=\pm 3$.

4. $x=\pm 10$.

5. $x=\pm 1$.

6. $x=\pm 4$.

1. $x=\pm 5$.

2. $x=\pm 10$.

3. $x=\pm 7$.

4. $x=\pm 9$.

Page 197.

5. $y=\pm 2$.

6. $y=\pm 5$.

7. $x=\pm 7$.

8. $x=\pm 10$.

9. $x=\pm 6$.

10. $x=\pm 8$.

11. $x=\pm 6$.

12. $x=\pm 2$.

13. $y=\pm 3$.

14. $x=\pm 6$.

15. $x=\pm 4$.

16. $x=\pm 3$.

17. $x=\pm 2$.

18. $x=\pm 6$.

19. $x=\pm 5$.

20. $x=\pm \frac{1}{2}$.

Page 197.

21. $x=\pm 2$.

22. $x=\pm 9$.

23. $x=\pm 6$.

Page 198.

1. $x=7$; $y=5$.

2. $x=8$; $y=5$.

3. $x=7$; $y=1$.

4. $x=3$; $y=2$.

Page 200.

1. $x=7$ or -4; $y=4$ or -7.

2. $x=6$ or 2; $y=2$ or 6.

3. $x=10$ or -2; $y=2$ or -10.

4. $x=9$ or -6; $y=2$ or -3.

Page 201.

1. $x=\pm 6$ or ± 3; $y=\pm 3$ or ± 6.

2. $x=\pm 10$ or ± 6; $y=\pm 3$ or ± 5.

3. $x=\pm \frac{9}{2}$ or $\pm \frac{3}{2}$; $y=\pm \frac{9}{2}$ or $\pm \frac{27}{2}$.

4. $x=\pm 20$ or ± 7; $y=\pm 7$ or ± 20.

Page 202.

1. $a+b$.

2. $a+4$.

3. Not a square.

4. $x+2y$.

5. $2x+2y$.

6. Not a square.

7. $x-3$.

8. $x+4$.

9. $x-9$.

Page 202.

10. Not a square.
11. $x+7$.
12. $x+\frac{1}{4}$.
13. $2x+y$.
14. $2x-3y$.
15. $y-\frac{5}{2}$.
16. $x+\frac{1}{2}$.

Page 203.

1. $x-y=\pm 5$.
2. 64.
3. ± 7.
4. 4.
5. ± 4.
6. 2.
7. 16.
8. 4.
9. 12.
10. 36.
11. ± 6.

Page 204.

12. 4.
13. 2.
14. 0.
15. 0.
16. 4.
17. $12xy$.
18. 1.
19. 4.

1. $x=6$ or 1; $y=1$ or 6.
2. $x=3$; $y=2$.
3. $x=\pm 7$ or ± 4; $y=\pm 4$ or ± 7.
4. $x=5$ or 3; $y=3$ or 5.

Page 205.

5. $x=\pm 3$ or ± 1; $y=\pm 1$ or ± 3.
6. $x=5$ or -2; $y=2$ or -5.
7. $x=\pm 3$; $y=\pm 4$.
8. $x=9$ or -2; $y=2$ or -9.
9. $x=\pm 7$ or ± 5; $y=\pm 5$ or ± 7.
10. $x=4$ or -2; $y=1$ or -2.
11. $x=4$ or 1; $y=3$ or 12.
12. $x=4$; $y=4$.
13. $x=5$ or -3; $y=3$ or -5.
14. $x=4$ or -2; $y=2$ or -4.

1. $10.
2. 16 acres; $48 per acre.

Page 206.

3. 8 rods and 6 rods.
4. 12 rods and 7 rods.
5. 32 yds. and 20 yds.
6. 100 ft. and 30 ft.
7. Length, 10 rods; price, $6 per rod.
8. 12 rods and 9 rods.
9. First, 10 yards at 10 cents; second, 6 yards at 14 cents.

Page 207.

1. 6 or 2.
2. 2 or -8.
3. -5 or -11.
4. 14 or 4.

Page 207.

5. 11 or 9.
6. 6 or -14.
7. -2 or -8.
8. 9 or 7.
9. 9 or -7.
10. -1 or -5.

Page 208.

1. $x-1=\pm 5$.
2. $x-4=\pm 7$.
3. $x-\frac{7}{2}=\pm\frac{9}{2}$.
4. $x-\frac{9}{2}=\pm\frac{11}{2}$.
5. Not a square.
6. $x+\frac{1}{2}=\pm\frac{3}{2}$.
7. Not a square.
8. $x+4=\pm 9$.
9. 9.
10. 4; $x^2-8x+16=25$.
11. 4.
12. 16; $x^2-8x+16=36$.
14. 36; $x^2+12x+36=49$.

Page 209.

1. 9.
2. 1.
3. 36.
4. 4.
5. $\frac{9}{4}$.
6. $\frac{1}{4}$.
7. 4.
8. 9.

Page 210.

1. $x=2$ or -4.
2. $x=\frac{1}{2}$ or $-\frac{5}{2}$.
3. $x=12$ or -2.
4. $x=8$ or 2.
5. $x=1$ or -7.

Page 210.

6. $x=2$ or 1.
7. $x=1$ or $-\frac{5}{3}$.
8. $x=\frac{5}{2}$ or -1.
9. $x=3$ or -4.
10. $x=3$ or -6.

Page 211.

1. $x=12$ or -6.
2. $x=2$ or -14.
3. $x=2$ or -12.
4. $x=2$ or -7.
5. $x=20$ or -4.
6. $x=-1$ or -3.
7. $x=\frac{3}{4}$ or $-\frac{11}{4}$.
8. $x=\frac{1}{2}$ or $-\frac{3}{2}$.
9. $x=6$ or -5.
10. $x=3$ or -4.
11. $x=3$ or -2.
12. $x=8$ or 4.
13. $x=4$ or -2.
14. $x=6$ or $-\frac{2}{3}$.
15. $x=7$ or 1.
16. $x=8$ or -2.
17. $x=6$ or 4.
18. $x=\frac{3}{2}$ or $\frac{2}{3}$.
19. $x=4$ or $-\frac{1}{2}$.
20. $x=9$ or -1.

Page 213.

1. $x=\frac{2}{3}$ or 1.
2. $x=3$ or $-\frac{13}{2}$.
3. $x=5$ or -9.
4. $x=\frac{1}{4}$ or $-\frac{1}{8}$.
5. $x=2$ or $-\frac{7}{6}$.
6. $x=3$ or $-2\frac{3}{7}$.
7. $x=1$ or -2.
8. $x=7$ or -1.
9. $x=3$ or -6.
10. $x=7$ or 4.
11. $x=4$ or $-\frac{7}{4}$.

Page 213.

12. $x=1$ or -5.
13. $x=4$ or $-\frac{1}{4}$.
14. $x=1\frac{2}{5}$ or $\frac{1}{5}$.
15. $x=\frac{5}{3}$ or $-\frac{5}{2}$.

1. Length, 10 rods; width, 6 rods.
2. 5 yards.
3. 20 yards, at 16 cents a yard.

Page 214.

4. 10.
5. 9.
6. 6 miles going; 8 returning.
7. 4 going; 3 returning.
8. 8.
9. Man, 4; son, 12.
10. A, 6 hours; B, 10 hours.
11. 6.
12. 4, 6, and 8 miles per hour.

Page 215.

1. $16a^2x^2$.
2. $81x^4y^6$.
3. $a^8b^{16}x^{12}$.
4. $c^{12}d^6x^6$.
5. $2x^2y$.
6. $3ax^2$.
7. x^2y.
8. a^2b^3.
9. $2c^4d$.
10. $3m^3n^2$.
11. $7a^3x^2$.
12. $2a^2xy^3$.

Page 216.

1. 12.
2. 35.
3. 16.
4. 10.
5. $7a$.
6. ab.
7. $4x$.
8. $12axy$.
9. $3b^3$.
10. $20x^2y^3$.
11. $9a^3$.
12. $8x^2y^2$.

1. 14.
2. 24.
3. $4ax^2$.
4. $4b^2$.
5. $10a^2x$.
6. $9c^2x$.

Page 217.

1. $9\sqrt{15}$.
2. $42\sqrt{3}$.
3. $16x\sqrt{b}$.
4. $ax^2\sqrt{y}$.
5. $6bc^2\sqrt{x}$.
6. $12axy^2\sqrt{y}$.
7. $10cd^2\sqrt{c}$.
8. $28b^5c\sqrt{3xy}$.

Page 218.

1. 4 and 5.
2. 25 and 2.
3. 64 and 1.
4. 100 and 3.
5. a^2 and a.
6. a^2x^2 and x.
7. a^4x^4 and a.
8. x^2y^2 and ax.
9. $9x^4y^4$ and y.

Page 218.

10. $4a^4y^2$ and $7a$.
11. $9b^2$ and $5bc$.
12. $36a^2b^2$ and $3ax$.
13. $\frac{1}{4}$ and 5.
14. $\frac{1}{9}$ and 7.
15. $\frac{4}{25}$ and 3.
16. $\frac{9}{49}$ and 2.
17. $\frac{1}{64}$ and 3.
18. $\frac{25}{36}$ and 2.

Page 219.

1. 8 and 2.
2. 27 and 4.
3. $8x^3$ and a.
4. $27c^3d^6$ and c.
5. $125y^3z^3$ and $2x^2y$.
6. $8b^3c^6$ and $6ab^2c^2$.

1. $2\sqrt{2}$.
2. $3\sqrt{2}$.
3. $10\sqrt{2}$.
4. $10\sqrt{7}$.
5. $12\sqrt{2}$.
6. $a\sqrt{x}$.
7. $xy\sqrt{y}$.
8. $3a^2x\sqrt{a}$.
9. $5b^2\sqrt{y}$.
10. $4x^2\sqrt{5}$.
11. $a\sqrt[3]{x^2}$.
12. $8a\sqrt[3]{a}$.
13. $35a^2x^2y\sqrt{3x}$.
14. $24xy\sqrt[3]{y}$.
15. $12y\sqrt{xy}$.

Page 220.

16. $10\sqrt[3]{4}$.
17. $24\sqrt{2}$.
18. $4a^2\sqrt{2b}$.
19. $6abc^2\sqrt{b}$.

Page 220.

20. $7abx\sqrt[3]{bx^2}$.
21. $4a^2x^2\sqrt{a}$.
22. $10x^2y\sqrt{2a}$.
23. $18axy^2\sqrt{6x}$.
24. $6y\sqrt[3]{2ay}$.
25. $8bc\sqrt{7b}$.
26. $\frac{1}{2}\sqrt{3}$.
27. $\frac{2}{3}\sqrt{2}$.
28. $\frac{1}{2}\sqrt{15}$.
29. $\frac{5}{2}\sqrt{2}$.
30. $\frac{5}{x}\sqrt{3}$.
31. $\frac{6}{b}\sqrt{a}$.
32. $ac\sqrt{c}$.
33. $\frac{12x}{5}$.

1. $\frac{1}{7}\sqrt{7}$.
2. $\frac{1}{5}\sqrt{5}$.
3. $\frac{1}{3}\sqrt{6}$.
4. $2\sqrt{2}$.
5. $\frac{2}{5}\sqrt{10}$.
6. $\frac{1}{7}\sqrt{35}$.
7. $\frac{5}{2}\sqrt{2}$.
8. $\frac{1}{2}\sqrt{6}$.
9. $\frac{7}{5}\sqrt{15}$.

Page 221.

1. $6\sqrt{5}$.
2. $3\sqrt{3}$.
3. $9\sqrt[3]{10}$.
4. $4\sqrt{7}$.
5. $8\sqrt{15}$.
6. $\sqrt[3]{ax}$.
7. $7\sqrt{2}+2\sqrt{5}$.
8. $2\sqrt[3]{10}+10\sqrt{10}$.
9. $4\sqrt{ax}-7\sqrt{2x}$.
10. $6\sqrt{5x}+4\sqrt{15}$.

Page 222.

1. $12\sqrt{6}$.
2. $9\sqrt{2}$.
3. $13\sqrt{5}$.
4. $8\sqrt[3]{2}$.
5. $2\sqrt{a}$.
6. $3a\sqrt{x}$.
7. $7\sqrt[3]{2a}$.
8. $23\sqrt{10}$.

Page 223.

9. $2\sqrt{3}$.
10. $9\sqrt{2}$.

1. $4\sqrt{2}$.
2. $9\sqrt{3}$.
3. $\sqrt{3}$.
4. $6\sqrt{3}$.
5. $5a\sqrt{x}$.
6. $8x\sqrt{5x}$.

1. $\sqrt{x^2y^2}$.
2. $\sqrt[3]{27a^3x^3}$.
3. $\sqrt{49a^6y^2}$.
4. $\sqrt[4]{a^4x^{12}}$.

Page 224.

1. $\sqrt{63}$.
2. $\sqrt{32}$.
3. $\sqrt{25a^2x}$.
4. $\sqrt[3]{8a^3y^2}$.
5. $\sqrt{36xy^2}$.
6. $\sqrt{50}$.

Page 225.

1. $\sqrt{a^6}=a^3$.
2. $\sqrt[3]{b^9}=b^3$.
3. $\sqrt{a^4b^6}=a^2b^3$.
4. $4\sqrt{a^{10}}=4a^5$.

Page 225.

5. $\sqrt[3]{x^{12}y^6}=x^4y^2$.
6. $\sqrt{a^4b^{10}c^6}=a^2b^5c^3$.

1. $\sqrt{a}$.
2. $\sqrt{b^3}$.
3. $\sqrt[3]{a^2b^5}$.
4. $7\sqrt[3]{a}$.
5. $5\sqrt{ab^3}$.
6. $4\sqrt[3]{x^7y^5}$.
7. $\sqrt[3]{5}$.
8. $\sqrt[3]{2a^2x^3}$.
9. $\sqrt{4xy^3}$.
10. $y^{\frac{1}{2}}$.
11. $3x^{\frac{1}{3}}$.
12. $a^{\frac{2}{4}}$.
13. $2x^{\frac{3}{2}}$.
14. $2^{\frac{1}{2}}$.
15. $6x^{\frac{2}{3}}y^{\frac{4}{3}}$.

Page 226.

1. $x^{\frac{4}{6}}$, $y^{\frac{3}{6}}$.
2. $x^{\frac{2}{4}}$, $x^{\frac{1}{4}}$.
3. $x^{\frac{3}{6}}$, $x^{\frac{2}{6}}$, $a^{\frac{1}{6}}$.
4. $a^{\frac{4}{12}}$, $b^{\frac{3}{12}}$.
5. $c^{\frac{3}{6}}$, $d^{\frac{1}{6}}$.
6. $a^{\frac{6}{4}}$, $b^{\frac{1}{4}}$.

1. $\sqrt[6]{125}$, $\sqrt[6]{9}$.
2. $\sqrt[6]{a^3x^9}$, $\sqrt[6]{x^4}$.
3. $4\sqrt[4]{a^3}$, $2\sqrt[4]{a^{10}}$.
4. $\sqrt[6]{x^4}$, $\sqrt[6]{a^3x^3}$,
$\sqrt[6]{xy}$.
5. $\sqrt[24]{a^{18}}$, $\sqrt[24]{a^3x^3}$,
$\sqrt[24]{a^8x^{16}}$.
6. $\sqrt[10]{2^{15}}$, $\sqrt[10]{3^{10}}$, $\sqrt[10]{5^6}$.

Page 227.

1. $3\sqrt{10}$.
2. $8\sqrt{21}$.
3. $3\sqrt[3]{6a^2x}$.
4. $2\sqrt{3}$.
5. $2\sqrt[3]{7}$.
6. $\sqrt{2y}$.
7. $40\sqrt{11}$.
8. $2\sqrt{7}$.
9. $2\sqrt[3]{4x}$.

Page 228.

1. $2\sqrt{15}$.
2. $8\sqrt{7}$.
3. $\sqrt[6]{72}$.
4. $3a\sqrt{2x}$.
5. $x\sqrt{ay}$.
6. $\sqrt[6]{a^3x^4y^5}$.
7. $15\sqrt{3}$.
8. $3\sqrt[6]{\frac{3ac^5}{4}}$.
9. $\sqrt[3]{3}$.
10. $\frac{1}{3}\sqrt[6]{\frac{5}{4}}$.
11. $4\sqrt[6]{a^4x}$.
12. $2\sqrt{a}$.

Page 229.

1. $3x$.
2. $4x$.
3. $6x$.
4. $4x$.
5. x^3.
6. x^4.
7. x^6.
8. x^4.

Page 231.

1. x.
2. x^{-2}.
3. $2x^2$.
4. $2x$.
5. $5x^{-2}y$.
6. $4x^{-1}y^{-3}$.

Page 231.

7. x^3y^{-4}.
8. $x^{-2}y^{-1}z^3$.
9. $3a^{-1}c^2x^{-3}$.
10. $x^{-3}y^2$.
11. a^5.
12. a^{-1}.
13. x^{-6}.
14. $a^{-5}x^4$.
15. x^2y^2.

Page 232.

1. $\frac{3}{x^2}$.
2. $\frac{4a}{y^4}$.
3. $\frac{2x}{y^2}$.
4. $\frac{8}{a^2b}$.
5. $\frac{9y^2}{x^4}$.
6. $\frac{10a^3}{c^2}$.
7. $\frac{12cd}{xy^3}$.
8. $\frac{5a^3c}{xy^3}$.
9. $\frac{m^2}{c^3n^3}$.

Page 233.

1. $\frac{5ab^2y^2}{3x^3}$.
2. $\frac{3a^4b^2n}{4m^3}$.
3. $\frac{6a^6}{b^5}$.
4. $\frac{bc^3x^4}{a^3y}$.
5. $\frac{a^2x^2}{y^2}$.

Page 233.

6. $\dfrac{mx^3}{a^3y^4}$.

1. $2a^2b^3x^2y$.
2. $3amn^4$.
3. $5ab^{-2}xy^{-1}$.
4. m^3x^5.
5. $2abc^2x^3y^3$.
6. $a^4bdm^{-1}n$.

Page 234.

1. $7a^{\frac{1}{2}}x^{-2}-a^2x^{-5}$.
2. $5a^{-2}b^3-5y^{-2}$.
3. $x^{-3}+x^{\frac{1}{2}}$.
4. $3x^{-2}y^2-5x^{-3}$
$-2x^3$.
5. $7ab-7ab^{-1}$
$+10a^3c^{-3}$.

1. a^5x^2.
2. $6cd^{-3}$.
3. $15a^2x^{-3}y^{-2}$.
4. $6a^3c^{-3}x^{-2}$.
5. $5a^4y^2x^{-2}$.
6. 1.
7. a^2x.
8. $4a$.
9. $5xy$.
10. $3a$.
11. ax.
12. $x^{\frac{5}{6}}$.
13. $x^{\frac{1}{2}}$.
14. $a^{\frac{2}{3}}$.
15. $x^{\frac{1}{3}}y^{\frac{1}{2}}$.
16. $x^2y^{-\frac{1}{3}}$.
17. $2c^{\frac{1}{2}}d^{\frac{1}{3}}$.
18. $2a^{\frac{1}{2}}y^{\frac{1}{2}}$.

Page 235.

19. y^6.
20. $2a^{\frac{1}{2}}x$.

1. $9\sqrt{a}+4\sqrt[3]{y}$.
2. $13x\sqrt{a}-5b^{-1}$
$-2a^3\sqrt{c}$.
3. $4b^{-1}\sqrt{x}-2\sqrt[3]{y^2}$.
$\sqrt{b}+2x^2y^{-1}$.
4. $2y\sqrt{x}-2x^2y$
$-6x\sqrt{y}$.
5. $a^2+a^{\frac{3}{2}}b^{\frac{1}{2}}+a^{\frac{1}{2}}b^{\frac{3}{2}}$
$+b^2$.
6. $x^{-1}+2+2x+x^2$.
7. $x^2+2x^{\frac{3}{2}}+2x+x^{\frac{1}{2}}$.
8. $2a^4+7a^2+5$
$+a^{-2}$.
9. $x^{\frac{5}{4}}+3x+3x^{\frac{3}{4}}+x^{\frac{1}{2}}$.
10. $a+2a^{\frac{1}{2}}b^{\frac{1}{2}}+b$.
11. $a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$.
12. $x^{-3}+2x^{-2}+2x^{-1}$.
13. $x+x^{\frac{5}{6}}+x^{\frac{2}{3}}$.
14. $x-x^{-1}$.

Page 236.

1. 3.
2. 7.
3. 5.
4. 13.
5. 1.
6. 13.
7. $x+y$.
8. $a(b+c)$.
9. $6\sqrt{a+b}$.
10. $2\sqrt{x^2-y^2}$.

Page 237.

11. $\sqrt{2x^2-2x}$.
12. $\sqrt{3x+12}$.
13. $\sqrt{a^2-b^2}$.

Page 237.

14. $\sqrt{x^2y-xy^2}$.
15. $\sqrt{a+x}$.
16. $a+b$.

1. 5.
2. 14.
3. $4x$.
4. $x+4$.
5. $45x-18$.
6. $\dfrac{x+1}{25}$.
7. $8x-\sqrt{x}$.
8. $4x-x^2$.
9. $2x-4$.

Page 238.

1. $x-4\sqrt{x}+4$.
2. $16-8\sqrt{3x}+3x$.
3. $4-12\sqrt{2x}+18x$.
4. $x-2\sqrt{x^2+2x}+x$
$+2$.
5. $x^2+2x\sqrt{x-4}+x$
-4.
6. $9x^2-6x\sqrt{9x^2-2x}$
$+9x^2-2x$.
7. $x=25$.
8. $x+1=9$.
9. $x=2x-8$.
10. $4x=x+4\sqrt{x}+4$.

Page 239.

1. $x=3$.
2. $x=4$.
3. $x=4$.
4. $x=7$.
5. $x=5$ or 1.
6. $x=6$.
7. $x=8$.
8. $x=4$ or 0.
9. $x=\pm 3$.

Page 239.

10. $x=\frac{12}{5}$
11. $x=5$ or -3.
12. $x=9$ or 6.
13. $x=6$ or 4.
14. $x=5$.

Page 240.

1. 2.
2. 4.
3. 5.
4. $\frac{1}{3}$.
5. $\frac{1}{5}$.
6. $\frac{1}{4}$.
7. $\frac{5}{8}$.
8. $\frac{6}{10}$ or $\frac{3}{5}$.
9. $\frac{4}{6}$ or $\frac{2}{3}$.
10. $\frac{8}{14}$ or $\frac{4}{7}$.
11. 3.
12. 3.
13. 7.
14. $\frac{1}{5}$.
15. $\frac{1}{8}$.
16. $\frac{1}{4}$.
17. $\frac{1}{3}$.
18. 2.
19. 12.
20. $\frac{1}{6}$.

1. $3 : 1 :: 2 : \frac{2}{3}$.
2. $4 : 6 :: 6 : 9$.

Page 240.

3. $2 : 8 :: 1 : 4$.
4. $5 : \frac{1}{2} :: 20 : 2$.

Page 241.

1. $9 : 3 :: 1 : \frac{1}{3}$.
2. $5 : 12 :: 10 : 24$.
3. $2 : 10 :: 3 : 15$.
4. $\frac{1}{2} : 4 :: 2 : 16$.
5. $6 : \frac{2}{3} :: 18 : 2$.

Page 242.

1. $1 : 3 :: 2 : 6$.
2. $a : b :: c : d$.
3. $3 : 4 :: x : y$.
4. $x+y : 3 :: x-2 : 4$.
5. $5y+1 : 6 :: 3 : 4$.
6. $\frac{2}{8}=\frac{3}{12}$.
7. $\frac{5}{3}=\frac{15}{9}$.
8. $\frac{a}{x}=\frac{b}{y}$.
9. $\frac{x+y}{x}=\frac{4}{3}$.
10. $\frac{x+3}{x-3}=\frac{3}{2}$.

1. $3x=5x-15$.
2. $7x=2y$.
3. $x+3=3x-15$.
4. $2x=3x-1$.
5. $2x+2y=3x-3y$.
6. $3x+15=35-5x$.

Page 243.

1. $x=3$.
2. $x=10$.
3. $x=-5$.
4. $x=2$; $y=8$.
5. $x=10$; $y=2$.
6. $x=7$; $y=4$.
7. $x=5$; $y=1$.
8. $x=4$; $y=3$;
   $z=1$.
9. $x=6$; $y=2$;
   $z=4$.
10. $x=6$; $y=2$.
11. $x=\pm 8$ or ± 6;
    $y=\pm 6$ or ± 8.
12. $x=17$.
13. $x=9$ or 8.
14. $x=5$ or $-\frac{3}{4}$.

Page 244.

1. 28 and 8.
2. 10.
3. 100 and 50.
4. \$4000, \$6000,
   \$8000.
5. 21 ounces.
6. A, 36; B, 45.
7. A, 900; B, 2100.
8. 2½ pounds.
9. 140 acres.

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